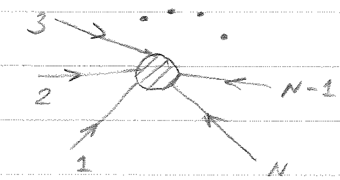
N-particle generalization of the Veneziano amplitude

- Search for a planar dual amplitude B_N in terms of all the planar energies $s_{ij} = (p_i + p_{i+1} + \ldots + p_j)^2$.

There are $\frac{1}{2}N(N-3)$ planar energies:



All momenta incoming

Proof: List them — e.g. $N=6$ case:

s_{12}				
s_{23}	s_{23}			
s_{14}	s_{24}	s_{34}		
$\cancel{s_{15}}$	s_{25}	s_{35}	s_{45}	
$\cancel{s_{16}}$	$\cancel{s_{26}}$	$\cancel{s_{36}}$	$\cancel{s_{46}}$	$\cancel{s_{56}}$

s_{16}: vanishes by mom. cons.

s_{15}: is just m^2 by mom cons.

Last row: These are already counted by mom. conservation.

e.g. $s_{14} = s_{56}$

$s_{13} = s_{46}$

$\vdots$

$$s_{16} = (p_1 + p_2 + \ldots + p_5 + p_6)^2 = 0$$

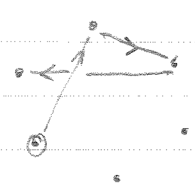
$$s_{15} = (p_1 + p_2 + p_3 + p_4 + p_5)^2 = (-p_6)^2 = m^2.$$

$$\boxed{\therefore \;\#\text{ of planar energies} = \underset{\text{First column}}{(N-3)} + \underbrace{\sum_{n=0}^{N-3} (N-3-n)}_{\text{rest of the columns}} = \frac{1}{2}N(N-3)}$$

The full amplitude is then constructed by adding up all inequivalent planar orderings of the planar dual planar amplitudes.

There are $\frac{1}{2}(N-1)!$ such orderings.

Proof: Number of cyclically inequivalent diagrams
= num. of unique ways to build a ring out of N vertices.



$$= \frac{1}{2}(N-1)(N-2)\cdots = \boxed{\frac{1}{2}(N-1)!}$$

$\frac{1}{2}$: Building in reverse yields the same ring.

Constructing planar dual amplitude

As in the $N=5$ case, we try:

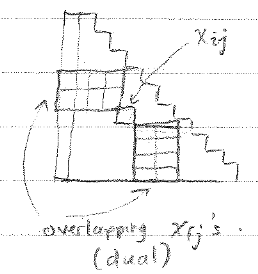
$$B_N(s_{12},\ldots) = \int_0^1 dx_{12}\ldots\ x_{12}^{-1-\alpha(s_{12})}\ldots f(x_{12},\ldots)$$

should prevent overlapping divergences.

Write down the $\frac{1}{2}N(N-3)$ constraint equations:

$$x_{ij} = 1 - \prod_{(kl)} x_{kl} = 1 - \prod_{k=1}^{i-1}\prod_{l=i}^{j-1} x_{kl} \prod_{m=i+1}^{j}\prod_{n=j+1}^{N-1} x_{mn}$$

(kl): over all overlapping variables



Looks impossible to solve, but it in fact a solution can be found.

There are $N-3$ redundant equations $\Rightarrow$ our integration variables

$\rightarrow$ Take them to be the variables in the first column of our list

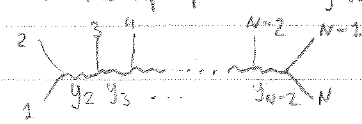
So, $f(x_{12},\ldots)$ is a product of $\frac{1}{2}(N-3)(N-2)$ delta functions.

Define:

$$y_2 = x_{12}$$
$$y_3 = x_{13}$$
$$\vdots$$
$$y_{N-2} = x_{1,N-2}$$

This is our choice – we picked the $N-3$ redundant variables.

$\rightarrow$ Turns out this corresponds to the multiperipheral configuration:



Then, upon solving the constraint equations, find all other variables x_{mn} are related to the y_i's by:

$$x_{mn} = \frac{(1-y_m y_{m+1}\cdots y_{n-1})(1-y_{m-1}\cdots y_n)}{(1-y_m y_{m+1}\cdots y_n)(1-y_{m-1}\cdots y_{n-1})} = \frac{\left(1-\prod_{i=m}^{n-1} y_i\right)\left(1-\prod_{i=m-1}^{n} y_i\right)}{\left(1-\prod_{i=m}^{n} y_i\right)\left(1-\prod_{i=m-1}^{n-1} y_i\right)}$$

with $y_1 = y_{N-1} = 0$ (they don't exist)

The Jacobian (without proof):

$$\left|\det\left(\frac{\partial C_i}{\partial x_j}\right)\right| = \prod_{j=2}^{N-3}(1-y_j y_{j+1})^{-1}$$

So, the integrations run over the y variables

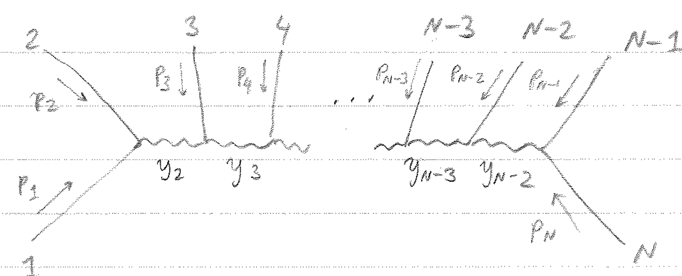
$$B_N = \int_0^1 dy_2 \cdots dy_{N-2} \prod_{(m,n)} x_{mn}(y)^{-1-\alpha(s_{mn})} \frac{1}{\left|\det\left(\frac{\partial c_i}{\partial x_j}\right)\right|}$$

$$= \int_0^1 dy_2 \cdots dy_{N-2} \prod_{(m,n)} x_{mn}(y)^{-1-\alpha(s_{mn})} \prod_{j=2}^{N-3} (1 - y_j y_{j+1})^{-1},$$

where $x_{1n} = y_n$ (convenient to pull out)

$$B_N = \int_0^1 dy_2 \cdots dy_{N-2} \prod_{i=2}^{N-2} y_i^{-1-\alpha(s_{1i})} \underbrace{\prod_{m=2}^{N-2} \prod_{n=m+1}^{N-1} x_{mn}(y)^{-1-\alpha(s_{mn})}}_{\cancel{\text{all channel variables}}\text{ except for those from first column.}} \prod_{j=2}^{N-3} (1 - y_j y_{j+1})^{-1}$$

$$\begin{array}{lllll}
y_2 & & & & \\
y_3 & x_{23} & & & \\
y_4 & x_{24} & x_{34} & & \\
\vdots & \vdots & \vdots & \ddots & \\
y_{N-2} & x_{2,N-2} & x_{3,N-2} & \cdots\ x_{N-3,N-2} & \\
 & x_{2,N-1} & x_{3,N-1} & x_{N-3,N-1} & x_{N-2,N-1}
\end{array}$$



Multi-peripheral configuration