Hiren H. Patel
hnpatel.net

Bardakçi–Ruegg form for N-particle dual amplitude

$$B_N = \int_0^1 dy_2 \cdots dy_{N-2} \prod_{i=2}^{N-2} y_i^{-1-\alpha(s_{1i})} \prod_{m=2}^{N-2} \prod_{n=m+1}^{N-1} x_{mn}(y)^{-1-\alpha(s_{mn})} \prod_{j=2}^{N-3} (1 - y_j y_{j+1})^{-1}$$

write out x_{mn} in terms of y_i's.

Recombine and express in terms of $p_i \cdot p_j$

– use $0 = \alpha' m^2 + \alpha_0$ ← comes from requiring external particle to lie on leading trajectory.

For $N=6$, the x_{mn}'s are as follows:

$$x_{23} = \frac{[1-y_2]}{1-y_2 y_3}$$

Box

$$x_{24} = \frac{1-y_2y_3}{1-y_2y_3y_4} \qquad x_{34} = \frac{[1-y_3](1-y_2y_3y_4)}{(1-y_3y_4)(1-y_2y_3)}$$

$$x_{25} = 1-y_2y_3y_4 \qquad x_{35} = \frac{1-y_3y_4}{1-y_2y_3y_4} \qquad x_{45} = \frac{[1-y_4]}{1-y_3y_4}$$

The recombinations split into three cases

1. $[1-y_i]$ one y
2. $(1-y_iy_{i+1})$ two y's
3. $(1-y_i \cdots y_{j-1})$ many y's (circled)

} involve BOXes of four x_{mn}'s.

Case ① $(1-y_i)^{-1-\alpha(s_{i,i+1})}$

$$\text{exponent} = -1 - [\alpha'(p_i + p_{i+1})^2 + \alpha_0]$$

$$= -1 - \underbrace{\alpha'(2m^2}\, + 2p_i\cdot p_{i+1}) - \alpha_0$$

$$= -1 + 2\alpha_0 - 2\alpha' p_i \cdot p_{i+1} - \alpha_0$$

$$= -1 + \alpha_0 - 2\alpha' p_i \cdot p_j \quad \text{where } j = i+1$$

recombination:

$$\Rightarrow \quad (1-y_i)^{-1+\alpha_0} (1-y_i)^{-2\alpha' p_i \cdot p_j} \qquad j = i+1$$

Hiren H. Patel
hhpatel.net

Case ② $(1-y_i y_{i+1})$

Jacobian

$$\text{exponent} = (-\cancel{1}-\alpha(s_{i,i+2})) - (-\cancel{1}-\alpha(s_{i,i+1})) - (-\cancel{1}-\alpha(s_{i+1,i+2})) - \cancel{1}$$

$$= \alpha'\left(-\underbrace{(p_i+p_{i+1}+p_{i+2})^2} + (p_i+p_{i+1})^2 + (p_{i+1}+p_{i+2})^2\right) - \alpha_0 + \alpha_0 + \alpha_0$$

$$= \alpha'\left(-\cancel{(p_i+p_{i+1})^2} - \underbrace{p_{i+2}^2} - 2(p_i+\cancel{p_{i+1}})p_{i+2} + \cancel{(p_i+p_{i+1})^2} + \underbrace{2m^2} + \cancel{2p_{i+1}\cdot p_{i+2}}\right) + \alpha_0$$

$$= \alpha'(+m^2 - 2p_i\cdot p_{i+2}) + \alpha_0 \quad = 0$$

$$= -2\alpha' p_i \cdot p_{i+2} \qquad \text{let } j = i+2$$

recombination:

$$\Longrightarrow \quad (1-y_i y_{j-1})^{-2\alpha' p_i\cdot p_j}$$

Case ③ $(1-y_i y_{i+1}\cdots y_{j-1})$

$$\text{exponent} = (-\cancel{1}-\alpha(s_{i,j})) - (-\cancel{1}-\alpha(s_{i,j-1})) - (-\cancel{1}-\alpha(s_{i+1,j})) + (-\cancel{1}-\alpha(s_{i+1,j-1}))$$

$$= \alpha'\left[-s_{i,j} + s_{i,j-1} + s_{i+1,j} - s_{i+1,j-1}\right] - \cancel{\alpha_0} + \cancel{\alpha_0} + \cancel{\alpha_0} - \cancel{\alpha_0}$$

$$= \alpha'\left[-\underbrace{(p_i+\dots+p_j)^2} + (p_i+\dots+p_{j-1})^2 + \underbrace{(p_{i+1}+\dots+p_j)^2} - (p_{i+1}+\dots+p_{j-1})^2\right]$$

$$= \alpha'\Big[\left\{-\cancel{(p_i+\dots+p_{j-1})^2} - \cancel{p_j^2} - 2(p_i+\dots+p_{j-1})\cdot p_j\right\} + \cancel{(p_i+\dots+p_{j-1})^2}$$

$$+ \left\{\cancel{(p_{i+1}+\dots+p_{j-1})^2} + \cancel{p_j^2} + 2(p_{i+1}+\dots+p_{j-1})p_j\right\} - \cancel{(p_{i+1}+\dots+p_{j-1})^2}\Big]$$

$$= \alpha'\left[-2(p_i+\cancel{p_{i+1}+\dots+p_{j-1}})\cdot p_j + 2(\cancel{p_{i+1}+\dots+p_{j-1}})\cdot p_j\right]$$

$$= -2\alpha' p_i\cdot p_j$$

recombination:

$$\Longrightarrow \quad (1-y_i y_{i+1}\cdots y_{j-1})^{-2\alpha' p_i\cdot p_j}$$

So, in total

$$\boxed{B_N = \int_0^1 \prod_{i=2}^{N-2}\left[dy_i (1-y_i)^{-1+\alpha_0}\, y_i^{-1-\alpha(s_{1i})}\right] \prod_{i=2}^{N-2}\prod_{j=i+1}^{N-1}(1-y_i y_{i+1}\cdots y_{j-1})^{-2\alpha' p_i\cdot p_j}}$$

Bardakçi – Ruegg