Anharmonic ratio

— also called cross-ratio or double-ratio.

$$(a,b;c,d) = \frac{(a-c)(b-d)}{(a-d)(b-c)} \qquad a,b,c,d \in \mathbb{C}, \text{ collinear or on a circle.}$$

Symmetries of ratio: Interchange any two pairs

$$= (b,a;d,c) = \frac{(b-d)(a-c)}{(b-c)(a-d)} \quad \checkmark$$

$$= (d,c;b,a) = \frac{(d-b)(c-a)}{(d-a)(c-b)} \quad \checkmark$$

$$= (c,d;a,b) = \frac{(c-a)(d-b)}{(c-b)(d-a)} \quad \checkmark$$

Reciprocal: Interchange first two, or last two numbers.

$$(a,b;c,d)^{-1} = \frac{(a-d)(b-c)}{(a-c)(b-d)}$$

$$= (a,b;d,c)$$

$$= (b,a;c,d) \quad \text{by symm.}$$

Sum rule: Interchange middle two numbers

$$(a,b;c,d) + (a,c;b,d) = \frac{(a-c)(b-d)}{(a-d)(b-c)} + \frac{(a-b)(c-d)}{(a-d)\underbrace{(c-b)}_{(-)}}$$

$$= \frac{ab - \cancel{ad} - \cancel{cb} + cd - ac + \cancel{ad} + \cancel{bc} - bd}{ab - ac - db + dc} = 1$$

$\therefore$ Although there are $4! = 24$ permutations of the arguments, the anharmonic ratio of four numbers can give only 6 distinct values:

$$\text{If } (a,b;c,d) = X$$

then $(a,c;b,d) = 1 - X$ (sum rule) $\qquad (a,b;d,c) = \frac{1}{X}$ (reciprocal)

$(a,c;d,b) = \frac{1}{1-X}$ (reciprocal on sum rule) $\qquad (a,d;b,c) = 1 - \frac{1}{X}$ (sum rule on reciprocal)

$$(a,d;c,b) = 1 - \frac{1}{1-X} = \frac{1}{1-\frac{1}{X}} \quad \left(\begin{array}{c}\text{sum rule on}\\ \text{reciprocal on}\\ \text{sum rule}\end{array}\right) = \left(\begin{array}{c}\text{reciprocal on}\\ \text{sum rule}\\ \text{on reciprocal}\end{array}\right).$$

Multiplication rules:

$$(a,b;c,d)\times(a,b;d,e) = \frac{(a-c)\cancel{(b-d)}}{\cancel{(a-d)}(b-c)}\,\frac{\cancel{(a-d)}(b-e)}{(a-e)\cancel{(b-d)}} = (a,b;c,e)$$

$$(a,b;c,d)\times(e,a;c,d) = \frac{\cancel{(a-c)}(b-d)}{\cancel{(a-d)}(b-c)}\,\frac{(e-c)\cancel{(a-d)}}{(e-d)\cancel{(a-c)}} = (b,e;\underbrace{d,c}_{\text{Flipped!}})$$

Invariance of the anharmonic ratio under Möbius transformations — Solution to exercise

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

$$\longrightarrow \frac{\left(\frac{az_1+b}{cz_1+d} - \frac{az_3+b}{cz_3+d}\right)\left(\frac{az_2+b}{cz_2+d} - \frac{az_4+b}{cz_4+d}\right)}{\left(\frac{az_1+b}{cz_1+d} - \frac{az_4+b}{cz_4+d}\right)\left(\frac{az_2+b}{cz_2+d} - \frac{az_3+b}{cz_3+d}\right)}$$

denominators same

$$= \frac{\left[(az_1+b)(cz_3+d) - (az_3+b)(cz_1+d)\right]_1 \left[(az_2+b)(cz_4+d) - (az_4+b)(cz_2+d)\right]_2}{\left[(az_1+b)(cz_4+d) - (az_4+b)(cz_1+d)\right]_3 \left[(az_2+b)(cz_3+d) - (az_3+b)(cz_2+d)\right]_4}$$

$$[\ldots]_1 = \cancel{acz_1z_3} + adz_1 + bcz_3 + \cancel{bd} - \cancel{acz_1z_3} - adz_3 - bcz_1 - \cancel{bd}$$

$$= (ad-bc)(z_1 - z_3)$$

$$[\ldots]_2 = adz_2 + bcz_4 - adz_4 - bcz_2$$

$$= (ad-bc)(z_2 - z_4)$$

$$[\ldots]_3 = adz_1 + bcz_4 - adz_4 - bcz_1$$

$$= (ad-bc)(z_1 - z_4)$$

$$[\ldots]_4 = adz_2 + bcz_3 - adz_3 - bcz_2$$

$$= (ad-bc)(z_2 - z_3)$$

$$= \frac{\cancel{(ad-bc)^2}(z_1 - z_3)(z_2 - z_4)}{\cancel{(ad-bc)^2}(z_1 - z_4)(z_2 - z_3)} \quad \checkmark$$