Coherent states

SHO: $[\hat{a}, \hat{a}^\dagger] = 1$

define coherent state: $|\alpha\rangle \equiv \mathcal{N} e^{\alpha \hat{a}^\dagger} |0\rangle = \mathcal{N} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$, $\alpha \equiv$ complex no.

h.c.

$$\langle\alpha| \equiv \langle 0| \mathcal{N} e^{\alpha^* \hat{a}} = \mathcal{N} \sum_{n=0}^{\infty} \langle n| \frac{(\alpha^*)^n}{\sqrt{n!}}$$

Properties

① $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ ($|\alpha\rangle$ e-state of $\hat{a}$, e-value $\equiv \alpha$)

— proven in CHEM 430 exam I

② $\beta^{\hat{a}^\dagger \hat{a}} |\alpha\rangle = |\alpha \times \beta\rangle$, $\beta \equiv$ complex number

proof:
$$\beta^{\hat{a}^\dagger\hat{a}} |\alpha\rangle = e^{(\ln\beta)\hat{a}^\dagger\hat{a}} |\alpha\rangle$$
$$= \left[\sum_{n=0}^{\infty} \frac{(\ln\beta)^n}{n!} (\hat{a}^\dagger\hat{a})^n\right]\left[\sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} |m\rangle\right]$$
$$= \sum_{n=0}^{\infty} \frac{(\ln\beta)^n}{n!} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} m^n |m\rangle$$
$$= \sum_{m=0}^{\infty} \left(\frac{\alpha^m}{\sqrt{m!}} |m\rangle \underbrace{\sum_{n=0}^{\infty} \frac{(\ln\beta)^n}{n!} m^n}_{e^{m\ln\beta} = \beta^m} \right)$$
$$= \sum_{m=0}^{\infty} \frac{(\alpha\beta)^m}{\sqrt{m!}} |m\rangle = |\alpha\beta\rangle$$
✓ (not normalized).

③ $\langle\beta|\alpha\rangle = \mathcal{N}_\beta^* \mathcal{N}_\alpha e^{\beta^* \alpha [a, a^\dagger]}$ ← coherent states not orthogonal (over-complete)

— proven in notes

④ $f(a)\beta^{\hat{a}^\dagger\hat{a}} = \beta^{\hat{a}^\dagger\hat{a}} f(\beta^{[a,a^\dagger]}\hat{a})$, $f(x) =$ function with T. series at $x = 0$

proof:
$$f(\hat{a})\beta^{\hat{a}^\dagger\hat{a}} = \left[\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \hat{a}^n\right]\left[\sum_{m=0}^{\infty} \frac{(\ln\beta)^m}{m!} (\hat{a}^\dagger\hat{a})^m\right]$$
$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{m=0}^{\infty} \frac{(\ln\beta)^m}{m!} \underbrace{\hat{a}^n (\hat{a}^\dagger\hat{a})^m}_{(n[a,a^\dagger] + \hat{a}^\dagger\hat{a})^m a^n}$$
(can be proven by induction)

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \underbrace{\sum_{m=0}^{\infty} \frac{(\ln\beta)^m}{m!} \left(n[a,a^\dagger] + \hat{a}^\dagger\hat{a}\right)^m}_{\beta^{[a,a^\dagger]}\,\beta^{\hat{a}^\dagger\hat{a}}} a^n$$

$$= \beta^{\hat{a}^\dagger\hat{a}} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \left(\beta^{[a,a^\dagger]}\right)^n \hat{a}^n = \beta^{\hat{a}^\dagger\hat{a}} f\left(\beta^{[a,a^\dagger]}\hat{a}\right) \quad \checkmark$$

⑤ $\beta^{\hat{a}^\dagger\hat{a}} f(\hat{a}^\dagger) = f\left(\beta^{[a,a^\dagger]} a^\dagger\right) \beta^{a^\dagger a}$ by h.c, and renaming $f^* \to f$, $\beta^* \to \beta$.

⑥ <u>Resolution of identity</u>

$$1 = \frac{1}{\pi}\int d^2\alpha\, e^{-|\alpha|^2} |\alpha\rangle\langle\alpha|$$ $d^2\alpha \equiv$ integration over entire complex plane.

<u>check:</u> $$= \frac{1}{\pi}\int d^2\alpha\, e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sum_{m=0}^{\infty} \frac{(\alpha^*)^m}{\sqrt{m!}} |n\rangle\langle m|$$ $\left(|N|^2 = 1\right)$

polar coord. $\alpha = \rho e^{i\theta}$, $\alpha^* = \rho e^{-i\theta}$

$$= \frac{1}{\pi}\int_0^{2\pi} d\theta \int_0^{\infty} d\rho\, \rho\, e^{-\rho^2} \sum_{n=0}^{\infty} \frac{\rho^n e^{in\theta}}{\sqrt{n!}} \sum_{m=0}^{\infty} \frac{\rho^m e^{-im\theta}}{\sqrt{m!}} |n\rangle\langle m|$$

$$= \frac{1}{\pi} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \underbrace{\int_0^{2\pi} d\theta\, e^{i(n-m)\theta}}_{(2\pi)\delta_{nm} \text{ for } n,m\in\mathbb{Z}} \int_0^{\infty} d\rho\, \rho\, e^{-\rho^2} \frac{\rho^{n+m}}{\sqrt{n!\,m!}} |n\rangle\langle m|$$

(Integral representation of kronecker delta)

$$= \frac{2\pi}{\pi} \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\int_0^{\infty} d\rho\, \rho^{2n+1} e^{-\rho^2}}_{\frac{1}{2}n! \text{ for } n\in\mathbb{Z}} |n\rangle\langle n|$$

$$= \sum_{n=0}^{\infty} |n\rangle\langle n| \quad \checkmark$$