Euler's gamma and beta functions

Representations of the Gamma function

$$\Gamma(z) = \int_0^\infty dt\, e^{-t}\, t^{-1+z}$$

Sterling's asymptotic series

$$\Gamma(z) \xrightarrow[z\to\infty]{} e^{-z} z^z \sqrt{2\pi z}\left(1+\frac{1}{z}+\dots\right)$$

$-\pi < \arg z < \pi$

At integer $z \le 0$, integration diverges in its lower limit

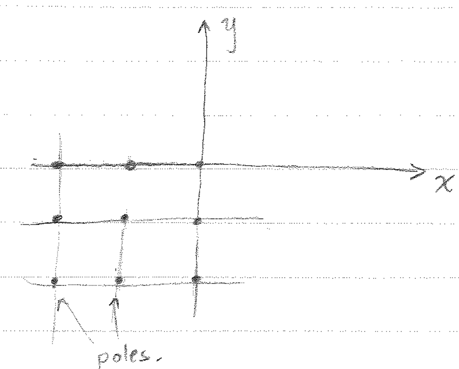
$\Rightarrow$ leads to poles. Can explicitly display these poles:

$$\Gamma(z) = \underbrace{\sum_{n=0}^\infty \frac{(-1)^n}{n!}\frac{1}{z+n}}_{\text{Poles of residue } (-1)^n/n!} + \underbrace{\int_1^\infty dt\, e^{-t} t^{-1+z}}_{\text{entire in } z.}$$

$\hookrightarrow$ Determines large z-behavior.

Representations of the Beta function

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$



Integral representation:

$$B(x,y) = \int_0^1 dt\, t^{-1+x}(1-t)^{-1+y}$$

At integer $x \le 0$, integration diverges in lower limit

At integer $y \le 0$, integration diverges in upper limit

<u>BUT</u> at integer $x = y \le 0$, get a pole in either x or y, but not both.

Can isolate poles —see next page.

$$B(x,y) = \sum_{n=0}^\infty \frac{(-1)^n}{n!}\frac{\Gamma(y)}{\Gamma(y-n)}\frac{1}{n+x}$$

can always exchange $x \leftrightarrow y$, due to symmetry of the function.

$$\equiv \sum_{n=0}^\infty \frac{1}{n!}\underbrace{(1-y)(2-y)\dots(n-y)}_{\text{Polynomial in } y\text{, of degree } n.}\frac{1}{n+x}$$

The Beta function is purely a sum of poles with no additional entire part.

Isolating the poles of Euler Beta function

$$B(x,y) = \int_0^1 dt \; t^{-1+x} \underbrace{(1-t)^{-1+y}}_{\text{write as a series around } t\approx 0}$$

$$(1-t)^{-1+y} \approx 1 + \frac{1}{1!}(-1)(-1+y)t + \frac{1}{2!}(-1)^2(-1+y)(-2+y)t^2 + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(-1+y)!}{(-1-n+y)!} t^n$$

expressed as Gamma functions for non-integer y. STILL a polynomial in y of order n.

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(y)}{\Gamma(y-n)} t^n$$

$$B(x,y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(y)}{\Gamma(y-n)} \int_0^t dt \; t^{-1+x} t^n$$

$$= \quad '' \quad \left[\frac{1}{n+x} t^{n+x}\right]_0^1$$

$$= \underline{\underline{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(y)}{\Gamma(y-n)} \frac{1}{n+x}}}$$

Usually, one introduces a residue function $R_n(y)$:

$$B(x,y) = \sum_{n=0}^{\infty} \frac{R_n(y)}{n+x}, \quad \text{where} \quad R_n(y) = \frac{(-1)^n}{n!} \frac{\Gamma(y)}{\Gamma(y-n)}$$

$$= \frac{(-1)^n}{n!} (-1+y)(-2+y)\ldots(-n+y)$$

$$\boxed{R_n(y) = \frac{1}{n!}(1-y)(2-y)\ldots(n-y)}$$

Notice that the residue is a polynomial in y of order n