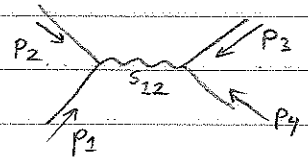
Residues in operator Formalism

Consider 2 → 2 scattering (with only one planar ordering)



$$s = (p_1 + p_2)^2 = s_{12}$$

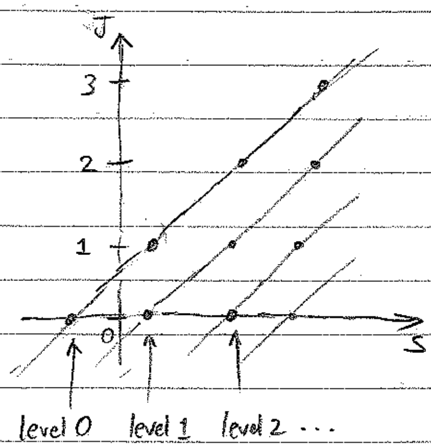
$$t = (p_2 + p_3)^2 = s_{23}$$

plus! (because momenta are incoming)

Then factorized form reads:

$$B_4(s,t) = \sum_{\{r\}} \sum_{r_0=0}^{\infty} (-1)^{r_n^\mu} \binom{r_0 - \alpha_0}{r_0} \langle 0 | \hat{V}(p_3) | \{r\} \rangle \frac{1}{-\alpha(s) + \bar{r} + r_0} \langle \{r\} | \hat{V}(p_2) | 0 \rangle$$

$$= \sum_{\{r\}} \sum_{r_0=0}^{\infty} (-1)^{r_n^\mu} \binom{r_0 - \alpha_0}{r_0} \frac{\langle 0 | e^{-\sqrt{2\alpha'}\, p_3 \cdot \sum \frac{\hat{a}_n}{\sqrt{n}}} | \{r\} \rangle \langle \{r\} | e^{\sqrt{2\alpha'}\, p_2 \cdot \sum \frac{a_n^\dagger}{\sqrt{n}}} | 0 \rangle}{-\alpha(s) + \bar{r} + r_0}$$



Where $\bar{r} = \sum_{n=1}^{\infty} n r_n^\mu$ is weighted sum in list.

and $N = \bar{r} + r_0$ is mass level.

(n.b. $r_0 = 0$ for unit intercept $\alpha_0 = 1$)

Consider $N=0$ level.

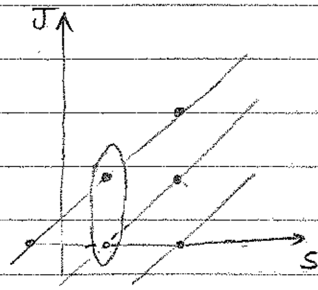
Only possible partition is $r_0 = 0$, $\{r_1^\mu = r_2^\mu = \ldots = 0\}$.

We have $(-1)^{r_n^\mu} \binom{r_0 - \alpha_0}{r_0} = +1$

and $\langle 0 | \hat{V} | \{0\} \rangle \langle \{0\} | \hat{V} | 0 \rangle = 1$

$$\Rightarrow \text{residue} = \underline{\underline{1}} = P_0(z) \quad ✓$$

Next, consider $N=1$ level.



Possible partitionings

1. $r_0 = 0, \quad r_1^\mu = 1 \quad \{r_2^\mu = r_3^\mu = \dots = 0\}.$

(2. $r_0 = 1 \quad \{r_1^\mu = r_2^\mu = \dots = 0\}.$)

Expect $J=0$ & $J=1$ states.

$\boxed{\text{Partition 1}}$ $\quad r_0 = 0, \quad r_1^\mu = 1$

We have $(-1)^{r_m^\mu} \binom{r_0 - \alpha_0}{r_0} = -1$ $\qquad (*)$

Then Taylor expand $e^{-\sqrt{2\alpha'}\, p_3 \cdot \sum \frac{\hat{a}_n}{\sqrt{n}}} = 1 - \sqrt{2\alpha'}\, p_3 \cdot \sum \frac{\hat{a}_n}{\sqrt{n}} + \dots$

to obtain matrix element:

$$\sum_{\mu=0}^{3} \langle 0 | e^{-\sqrt{2\alpha'}\, p_3 \cdot \sum \frac{a_n}{\sqrt{n}}} a_1^{\dagger\mu} | 0 \rangle \langle 0 | a_{1\mu} e^{+\sqrt{2\alpha'}\, p_2 \cdot \sum \frac{a_n^\dagger}{\sqrt{n}}} | 0 \rangle$$

$$= \sum_{\mu=0}^{3} \langle 0 | -\sqrt{2\alpha'}\, p_3 \cdot \sum_n \frac{a_n}{\sqrt{n}} a_1^{\dagger\mu} | 0 \rangle \langle 0 | a_{1\mu} \sqrt{2\alpha'}\, p_2 \cdot \sum_n \frac{a_n^\dagger}{\sqrt{n}} | 0 \rangle$$

only $n=1$ term survives in sum.

$$= -2\alpha' \sum_{\mu=0}^{3} p_{3\rho} p_2^\sigma \langle 0 | \underbrace{a_1^\rho a_1^{\dagger\mu}}_{-g^{\rho\mu}} | 0 \rangle \langle 0 | \underbrace{a_{1\mu} a_{1\sigma}^\dagger}_{-g_{\mu\sigma}} | 0 \rangle$$

$$= -2\alpha'\, p_3 \cdot p_2$$

In terms of COM scattering angle: $z = \cos\theta$ (part of exercise)

$$= -2\alpha' \left[\frac{-1}{4\alpha'} (1-z)(3\alpha_0 + 1) + \frac{\alpha_0}{\alpha'} \right]$$

$$= \frac{1}{2}(1-z)(3\alpha_0 + 1) - 2\alpha_0 \qquad (**)$$

So partial residue is (*) times (**)

$$\mathrm{Res}_{N=1}^{\text{part. 1}} = -\frac{1}{2}(1-z)(3\alpha_0+1) + 2\alpha_0$$

$\boxed{\text{Partition } 2}$ $r_0 = 1, \quad r_1^\mu = 0$

(non-contributing if $\alpha_0 = 1$)

We have $(-1)^{r_n^\mu}\binom{r_0-\alpha_0}{r_0} = +(1-\alpha_0)$

and $\langle 0|\hat{V}|\{0\}\rangle\langle\{0\}|\hat{V}|0\rangle = 1$

So partial residue is:

$$\mathrm{Res}_{N=1}^{\text{part 2}} = 1-\alpha_0$$

Thus the full residue at $N=1$ level is:

$$R_{N=1} = \mathrm{Res}_{N=1}^{\text{Part 1}} + \mathrm{Res}_{N=2}^{\text{Part 2}}$$

$$= -\frac{1}{2}(1-z)(3\alpha_0+1) + 2\alpha_0 + (1-\alpha_0)$$

organize by powers of z:

$$= \frac{1}{2}(3\alpha_0+1)P_1(z) + \frac{1}{2}(1-\alpha_0)P_0(z) \quad \checkmark$$

Exercise:

1. Show $p_3\cdot p_2 = \frac{-1}{4\alpha'}(1-z)(3\alpha_0+N) + \alpha_0$.

2. Compute residue of pole at $N=2$ level using operator formalism. Verify that result agrees with residue of Veneziano amplitude.

Solution to exercises

1. $t = (p_2+p_3)^2 = 2m^2 + 2\,p_2\cdot p_3$

$\Rightarrow \quad p_2\cdot p_3 = \frac{1}{2}t - m^2$

Equal mass kinematics: $t = -\frac{1}{2}(1-z)(s-4m^2)$

$= \frac{1}{2}\left[-\frac{1}{2}(1-z)(s-4m^2)\right] - m^2$

$\alpha(s) = \alpha' s + \alpha_0 = N \quad \Rightarrow \quad s = m^2\left(1-\frac{N}{\alpha_0}\right)$

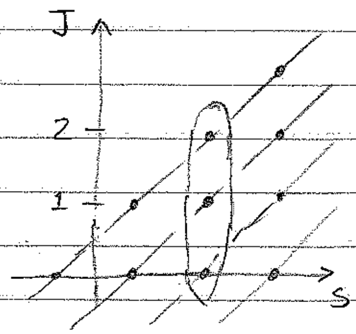
$= \frac{1}{2}\left[-\frac{1}{2}(1-z)\left(m^2 - \frac{m^2 N}{\alpha_0} - 4m^2\right)\right] - m^2$

$= +\frac{1}{4}(1-z)\,m^2\left(3+\frac{N}{\alpha_0}\right) - m^2$

Bootstrap condition: $m^2 = \frac{-\alpha_0}{\alpha'}$

$$\boxed{p_2\cdot p_3 = -\frac{1}{4\alpha'}(1-z)(3\alpha_0+N) + \alpha_0}$$

2. Residue at $N=2$ level



Possible partitionings

1. $r_0=0, \quad r_1=0, \quad r_2=1 \quad \{r_3^\mu = r_4^\mu = \ldots = 0\}$
2. $r_0=0, \quad r_1=2, \quad \{r_2^\mu = r_3^\mu = \ldots = 0\}$
3. $r_0=1, \quad r_1=1, \quad \{r_2^\mu = r_3^\mu = \ldots = 0\}$
4. $r_0=2, \quad \{r_1^\mu = r_2^\mu = \ldots = 0\}$

$\boxed{\text{Partition 1}}$ $\quad r_0=0, \quad r_2^\mu = 1$

then $(-1)^{r_2^\mu}\binom{r_0-\alpha_0}{r_0} = -1$

and $\sum_{\mu=0}^{3} \langle 0|\, e^{-\sqrt{2\alpha'}\,p_3\cdot\sum\frac{a_n}{\sqrt{n}}}\, \hat{a}_2^{\dagger\mu}|0\rangle\langle 0|\, a_{2\mu}\, e^{+\sqrt{2\alpha'}\,p_2\cdot\sum\frac{a_n^\dagger}{\sqrt{n}}}|0\rangle$

$\longrightarrow$ follow same calculation as in $N=1$ level, but this time only $n=2$ term in sum connects.

$$\langle 0|\hat{V}\,\hat{a}_2^{\dagger\mu}|0\rangle\langle 0|a_{2\mu}\hat{V}|0\rangle$$

using (**) from level 1 calculation

$n=2$ term.

$$=\left(\frac{1}{\sqrt{2}}\right)^2(-2\alpha' p_2\cdot p_3)$$

$$=-\alpha' p_2\cdot p_3$$

$$=\frac{1}{4}(1-z)(3\alpha_0+1)-\alpha_0$$

$\Rightarrow$ partial residue is:

$$Res^{part\,1}_{N=2}=-\frac{1}{4}(1-z)(3\alpha_0+1)+\alpha_0$$

$\boxed{\text{Partition } 2}$ $\quad r_0=0 \quad r^\mu_{n=1}=2$

$$(-1)^{r^\mu_{n=1}}\binom{r_0-\alpha_0}{r_0}=1$$

Need to calculate:

$$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\langle 0|e^{-\sqrt{2\alpha'}\,p_3\cdot\sum\frac{a_n}{\sqrt{n}}}\underbrace{\frac{1}{\sqrt{2}}a_1^{\mu\dagger}a_1^{\nu\dagger}|0\rangle\langle 0|\frac{1}{\sqrt{2}}}_{\text{Fock state normalization}}a_{1\mu}a_{2\nu}e^{\sqrt{2\alpha'}\,p_2\cdot\sum\frac{a_n^\dagger}{\sqrt{n}}}|0\rangle$$

Need 2 powers of a:

$$e^{\square}\simeq\ldots\frac{1}{2!}(-\sqrt{2\alpha'})^2(p_{3\rho}\hat{a}_1^\rho)(p_{3\sigma}\hat{a}_1^\sigma)$$

$n=1$ term connects.

$$e^{\square}\simeq\ldots\frac{1}{2!}(\sqrt{2\alpha'})^2(p_{2\alpha}a_1^{\dagger\alpha})(p_{2\beta}a_1^{\dagger\beta})$$

Taylor

$$=\sum_\mu\sum_\nu\left(\frac{1}{2!}(\sqrt{2\alpha'})^2\right)^2\left(\frac{1}{\sqrt{2}}\right)^2 p_{3\rho}p_{3\sigma}p_{2\alpha}p_{2\beta}\,\langle 0|a_1^\rho a_1^\sigma a_1^{\mu\dagger}a_1^{\nu\dagger}|0\rangle\langle 0|a_{1\mu}a_{1\nu}a_1^{\dagger\alpha}a_1^{\dagger\beta}|0\rangle$$

two contractions each: $(-1)^2(g^{\sigma\mu}g^{\rho\nu}+g^{\rho\mu}g^{\sigma\nu})(-1)^2(\delta_\nu{}^\alpha\delta_\mu{}^\beta+\delta_\mu{}^\alpha\delta_\nu{}^\beta)$

$$=(g^{\sigma\beta}g^{\rho\alpha}+g^{\sigma\alpha}g^{\rho\beta}+g^{\rho\beta}g^{\sigma\alpha}+g^{\rho\alpha}g^{\sigma\beta})$$

$$=2(g^{\rho\alpha}g^{\sigma\beta}+g^{\sigma\alpha}g^{\rho\beta})$$

$$= 2\left(\frac{1}{2!}(\sqrt{2\alpha'})^2\right)^2 \frac{1}{2}\left((p_2\cdot p_3)^2 + (p_3\cdot p_2)^2\right)$$

$$= 2\left(\frac{1}{2!}(\sqrt{2\alpha'})^2\right)^2 (p_2\cdot p_3)^2$$

$$\equiv 2\alpha'^2 (p_2\cdot p_3)^2$$

$$Res^{\text{part 2}}_{N=2} = 2\cancel{\alpha'^2}\left[\frac{-1}{4\cancel{\alpha'}}(1-z)(3\alpha_0+\overset{N=2}{2}) + \frac{\alpha_0}{\cancel{\alpha'}}\right]^2$$

$\boxed{\text{Partition 3}}$ $r_0 = 1$ $r_1^\mu = 1$

(noncontributing if $\alpha_0 = 1$)

We have $(-1)^{r_n^\mu}\binom{r_0-\alpha_0}{r_0} = (-)(1-\alpha_0)$

and $\sum_{\mu=0}^{3}\langle 0|e^{-\sqrt{2\alpha'}\,p_3\cdot\sum\frac{a_n}{\sqrt{n}}}\,a_1^{\mu\dagger}|0\rangle\langle 0|a_{\mu 1}\,e^{\sqrt{2\alpha'}\,p_2\cdot\sum_n\frac{a_n^\dagger}{\sqrt{n}}}|0\rangle$

$$= -2\alpha' p_2\cdot p_3$$ Using (**) from level 1 calculation

$$= -2\cancel{\alpha'}\left[\frac{-1}{4\cancel{\alpha'}}(1-z)(3\alpha_0+2) + \frac{\alpha_0}{\cancel{\alpha'}}\right]$$

$$= \frac{1}{2}(1-z)(3\alpha_0+2) - 2\alpha_0$$

$\rightarrow$ partial residue is

$$Res^{\text{part 3}}_{N=2} = -(1-\alpha_0)\left[\frac{1}{2}(1-z)(3\alpha_0+2) - 2\alpha_0\right]$$

$\boxed{\text{Partition 4}}$ $r_0 = 2$ $\{r_1^\mu = r_2^\mu = \ldots = 0\}$

(noncontrib if $\alpha_0 = 1$)

Here, $(-1)^{r_n^\mu}\binom{r_0-\alpha_0}{r_0} = \frac{1}{2}(1-\alpha_0)(2-\alpha_0)$

and $\langle 0|\hat{V}|\{0\}\rangle\langle\{0\}|\hat{V}|0\rangle = 1$

$\Rightarrow$ partial residue is

$$Res^{\text{part 4}}_{N=2} = \frac{1}{2}(1-\alpha_0)(2-\alpha_0)$$

Thus, full residue at $N=2$ level is:

$$R_{N=2} = \frac{1}{8}(\alpha_0 - 2)\,\alpha_0 - \frac{1}{4}(\alpha_0 - 1)(2 + 3\alpha_0) + \frac{1}{8}(2 + 3\alpha_0)^2 z^2$$

$$\equiv \underbrace{\frac{1}{12}(6\alpha_0^2 + 3\alpha_0 + 2)P_0(z)}_{J=0} - \underbrace{\frac{1}{4}(\alpha_0 - 1)(3\alpha_0 + 2)P_1(z)}_{J=1} + \underbrace{\frac{1}{12}(3\alpha_0 + 2)^2 P_2(z)}_{J=2}.$$