Verification of operator representation

$$V(p^\mu) = e^{-\sqrt{2\alpha'}\, p\cdot \sum_n \frac{a_n^\dagger}{\sqrt{n}}}\; e^{+\sqrt{2\alpha'}\, p\cdot \sum_n \frac{a_n}{\sqrt{n}}}$$

$$D(s) = \int_0^1 dy\; (1-y)^{-1+\alpha_0}\; y^{-1-\alpha(s)}\; y^{\sum_n (-n a_n^\dagger \cdot a_n)}$$

$$B_N = \langle 0 | \hat{V}(p_{N-1})\, \hat{D}(s_{1,N-1}) \cdots \hat{V}(p_3)\, \hat{D}(s_{12})\, \hat{V}(p_2) | 0 \rangle$$

$$= \int_0^1 \prod_{i=2}^{N-2} \left[dy_i\, (1-y_i)^{-1+\alpha_0}\, y_i^{-1-\alpha(s_{1i})} \right]$$

$$\times \langle 0 | \hat{V}(p_{N-1})\, y_{N-2}^{\sum(-n a_n^\dagger \cdot a_n)} \cdots y_3^{\sum(-n a_n^\dagger a_n)}\, \hat{V}(p_3)\, y_2^{\sum(-n a_n^\dagger a_n)}\, \hat{V}(p_2) | 0 \rangle$$

Now commute all the $y_i^{\sum(-n a_n^\dagger \cdot a_n)}$ to the right using
$$\begin{cases} y^{a^\dagger a} f(a^\dagger) = f(y^{[a,a^\dagger]} a^\dagger)\, y^{a^\dagger a} \\ y^{a^\dagger a} f(a) = f(y^{-[a,a^\dagger]} a)\, y^{a^\dagger a} \end{cases}$$

$$B_N = \int_0^1 \prod_{i=2}^{N-2} \left[dy_i\, (1-y_i)^{-1+\alpha_0}\, y_i^{-1-\alpha(s_{1i})} \right]$$

$$\times \langle 0 |\, e^{-\sqrt{2\alpha'}\, p_{N-1}\cdot \sum \frac{a_n^\dagger}{\sqrt{n}}}\; e^{+\sqrt{2\alpha'}\, p_{N-1} \sum \frac{a_n}{\sqrt{n}}}$$

$$e^{-\sqrt{2\alpha'}\, p_{N-2} \sum \frac{1}{y_{N-2}^n} \frac{a_n^\dagger}{\sqrt{n}}}\; e^{+\sqrt{2\alpha'}\, p_{N-2}\cdot \sum y_{N-2}^n \frac{a_n}{\sqrt{n}}}$$

$$\vdots$$

$$e^{-\sqrt{2\alpha'}\, p_3\cdot \sum \frac{1}{(y_3\cdots y_{N-2})^n} \frac{a_n^\dagger}{\sqrt{n}}}\; e^{+\sqrt{2\alpha'}\, p_3\cdot \sum (y_3\cdots y_{N-2})^n \frac{a_n}{\sqrt{n}}}$$

$$e^{-\sqrt{2\alpha'}\, p_2\cdot \sum \frac{1}{(y_2\cdots y_{N-2})^n} \frac{a_n^\dagger}{\sqrt{n}}}\; e^{+\sqrt{2\alpha'}\, p_2\cdot \sum (y_2\cdots y_{N-2})^n \frac{a_n}{\sqrt{n}}}$$

$$(y_2 y_3 \cdots y_{N-2})^{\sum(-n a_n^\dagger a_n)} | 0 \rangle \qquad (*)$$

Now $(y_2 y_3 \cdots y_{N-2})^{\sum(-n a_n^\dagger a_n)} = 1 + \ln(y_2 y_3 \cdots y_{N-2}) \sum_n (-n a_n^\dagger \cdot a_n) + \ldots$

so, acting on $|0\rangle$, only this term survives.

Continue to normal order.

Use Wick's theorem for exponentiated ladder operators.

Recall $\underbrace{f(a,a^\dagger)}_{\substack{\text{monomial in } a,\, a^\dagger \\ \text{eg: } a a^\dagger a a^\dagger}} = :f(a,a^\dagger): + \sum_{\substack{\text{single}\\ \text{contraction}}} :f(a,a^\dagger): + \sum_{\substack{\text{double}\\ \text{contraction}}} :f(a^\dagger,a): + \ldots$

Now: $\underbrace{g(a,a^\dagger)}_{\substack{\text{exponential of } a \,\&\, a^\dagger \\ \text{e.g: } (e^a e^{a^\dagger} e^a e^{a^\dagger})}} = :g(a,a^\dagger): \prod_{\substack{\text{all possible}\\ \text{contractions}}} g(a,a^\dagger)$

What is the contraction in our case? use $e^a e^{a^\dagger} = e^{a^\dagger} e^a e^{[a,a^\dagger]}$

$$e^{\sqrt{2\alpha'}\, p_{i\mu} \sum_n (y_i \cdots y_{N-2})^n \frac{a_n^\mu}{\sqrt{n}}}\; e^{-\sqrt{2\alpha'}\, p_{j\nu} \sum_{n'} \frac{1}{(y_j \cdots y_{N-2})^{n'}} \frac{a_{n'}^{\nu\dagger}}{\sqrt{n'}}}$$

$$= \underbrace{e^{\boxed{}\, a^\dagger}\; e^{\boxed{}\, a}}_{\text{Normal ordered}}\; e^{-2\alpha'\, p_{i\mu} p_{j\nu} \sum_{n,n'} \frac{(y_i \cdots y_{N-2})^n}{(y_j \cdots y_{N-2})^{n'}} \frac{[a_n^\mu,\, a_{n'}^{\nu\dagger}]}{\sqrt{nn'}}} \leftarrow -g^{\mu\nu}\delta_{nn'}$$

$-g^{\mu\nu}$

$$= \begin{pmatrix}\text{normal}\\ \text{ordered}\end{pmatrix} e^{+2\alpha'\, p_i \cdot p_j \sum_n \frac{(y_i \cdots y_{j-1})^n}{n}}$$

$$= \begin{pmatrix}\text{normal}\\ \text{ordered}\end{pmatrix} e^{2\alpha'\, p_i \cdot p_j \left[-\ln(1 - y_i \cdots y_{j-1})\right]}$$

$$= \begin{pmatrix}\text{normal}\\ \text{ordered}\end{pmatrix} \underbrace{(1 - y_i \cdots y_{j-1})^{-2\alpha' p_i \cdot p_j}}_{\text{the contraction.}}$$

Now, "all contractions" $\equiv$ $i<j$, such that $i \geq 2$ and $j \leq N-1$

$$\Rightarrow \quad \langle 0 | \ldots | 0 \rangle = \langle 0 | \begin{pmatrix}\text{Normal}\\ \text{ordered}\end{pmatrix} | 0 \rangle \prod_{2 \leq i < j \leq N-1} (1 - y_i \cdots y_{j-1})^{-2\alpha' p_i \cdot p_j}$$

So, finally,

$$\boxed{B_N = \int_0^1 \prod_{i=2}^{N-2} \left[dy_i\, (1-y_i)^{-1+\alpha_0}\, y_i^{-1-\alpha(s_{1i})} \right] \prod_{i=2}^{N-2} \prod_{j=i+1}^{N-1} (1 - y_i \cdots y_{j-1})^{-2\alpha' p_i \cdot p_j}}$$

Bardakçi-Ruegg