Cyclic symmetry

The operator Koba-Nielsen form displays cyclic symmetry:

$$B_N = \int_{-\infty}^{\infty} dz_1 \ldots dz_N \, |z_a - z_b||z_b - z_c||z_c - z_a| \prod_{i=1}^{N} |z_{i+1} - z_i|^{-1+\alpha_0}$$

$$\times \langle 0;0| \hat{V}(p_N, z_N)\hat{V}(p_{N-1}, z_{N-1}) \ldots \hat{V}(p_1, z_1)|0;0\rangle$$

① Commute $V(p_1, z_1)$ through to the left:

Need: $V(p_j, z_j)V(p_i, z_i) = \left(e^{Q_j} e^{P_j} e^{a_j^\dagger} e^{a_j}\right)\left(e^{Q_i} e^{P_i} e^{a_i^\dagger} e^{a_i}\right)$

$$= V(p_i, z_i)V(p_j, z_j)\, e^{[Q_j, P_i]} e^{[P_j, Q_i]} e^{[a_j^\dagger, a_i]} e^{[a_j, a_i^\dagger]}$$

commutators where calculated in previous exercise.

$$[P_j, Q_i] = -2\alpha' p_i \cdot p_j \ln z_j \qquad [a_j, a_i^\dagger] = -2\alpha' p_i \cdot p_j \ln\left(1 - \frac{z_i}{z_j}\right)$$

$$[Q_j, P_i] = +2\alpha' p_i \cdot p_j \ln z_i \qquad [a_j^\dagger, a_i] = +2\alpha' p_i \cdot p_j \ln\left(1 - \frac{z_j}{z_i}\right)$$

so: $V(p_j, z_j)V(p_i, z_i) = V(p_i, z_i)V(p_j, z_j)\, e^{-2\alpha' p_i \cdot p_j \left(\ln z_j - \ln z_i + \ln\left(1 - \frac{z_i}{z_j}\right) - \ln\left(1 - \frac{z_j}{z_i}\right)\right)}$

$$= V(p_i, z_i)V(p_j, z_j)\, e^{-2\alpha' p_i \cdot p_j \ln\left(\frac{z_j - z_i}{z_i - z_j}\right)}$$

$$= V(p_i, z_i)V(p_j, z_j)\, e^{-2\alpha' p_i \cdot p_j \ln(-1)}$$

$$\underline{V(p_j, z_j)V(p_i, z_i) = V(p_i, z_i)V(p_j, z_j)(-1)^{-2\alpha' p_i \cdot p_j}}$$

Thus,

$$\langle 0;0| \hat{V}(p_N, z_N) \ldots \hat{V}(p_1, z_1)|0;0\rangle$$

$$= \langle 0;0| \hat{V}(p_1, z_1)\hat{V}(p_N, z_N) \ldots V(p_2, z_2)|0;0\rangle (-1)^{-2\alpha' p_1 \cdot \underbrace{(p_2 + \ldots p_N)}_{-p_1 \text{ by mom cons.}}}$$

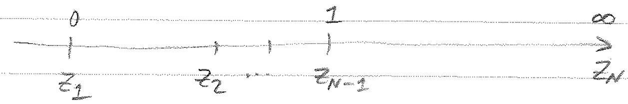
$$p_1^2 = m^2 = -\frac{\alpha_0}{\alpha'} \quad \text{(bootstrap)}$$

$$= \langle 0;0| \hat{V}(p_1, z_1)\hat{V}(p_N, z_N) \ldots V(p_2, z_2)|0;0\rangle \underbrace{(-1)^{-2\alpha_0}}_{\text{extra factor}}$$

② Integration ranges need to be adjusted.

Start by fixing z's so that the amplitude is in multiperipheral configuration:

$$z_1 = 0\,,\quad z_{N-1} = 1\,,\quad z_N = \infty$$



Make a change of integration variables:

$$z_1 = z_N\,,\quad z_2 = z_1\,,\ \ldots\ z_{N-1} = z_{N-2}\,,\quad z_N = z_{N-1}$$

The various $|z_{i+1} - z_i|$ factors need careful treatment; removing |abs|.

because $z_N > z_1$

$$\prod_{i=1}^{N} |z_{i+1} - z_i|^{-1+\alpha_0} = (z_2 - z_1)^{-1+\alpha_0} (z_3 - z_2)^{-1+\alpha_0} \cdots (z_N - z_{N-1})^{-1+\alpha_0} (z_1 - z_N)^{-1+\alpha_0} (-1)^{-1+\alpha_0}$$

$$\xrightarrow{\text{ch. var.}} (z_1 - z_N)^{-1+\alpha_0} (z_2 - z_1)^{-1+\alpha_0} \cdots (z_{N-1} - z_{N-2})^{-1+\alpha_0} (z_N - z_{N-1})^{-1+\alpha_0} (-1)^{-1+\alpha_0}$$

$$= |z_1 - z_N|^{-1+\alpha_0} |z_2 - z_1|^{-1+\alpha_0} \cdots |z_{N-1} - z_{N-2}|^{-1+\alpha_0} |z_N - z_{N-1}|^{-1+\alpha_0} (-1)^{2(-1+\alpha_0)}$$

because $z_{N-1} > z_N$

$$= \prod_{i=1}^{N} |z_{i+1} - z_i|^{-1+\alpha_0} \underbrace{(-1)^{2(-1+\alpha_0)}}_{\cancel{(-1)^{-2}}\,\underbrace{(-1)^{2\alpha_0}}_{\text{extra factor.}}}$$

$$|z_a - z_b||z_b - z_c||z_c - z_a| = (z_1 - z_{N-1})(z_{N-1} - z_N)\cancel{(-1)}^2 (z_N - z_1)$$

$$\xrightarrow{\text{ch. var.}} (z_N - z_{N-2})(z_{N-2} - z_{N-1})(z_{N-1} - z_N)$$

$$= |z_N - z_{N-2}||z_{N-2} - z_{N-1}|\cancel{(-1)^2}|z_{N-1} - z_N| = |z_a - z_b||z_b - z_c||z_c - z_a|$$

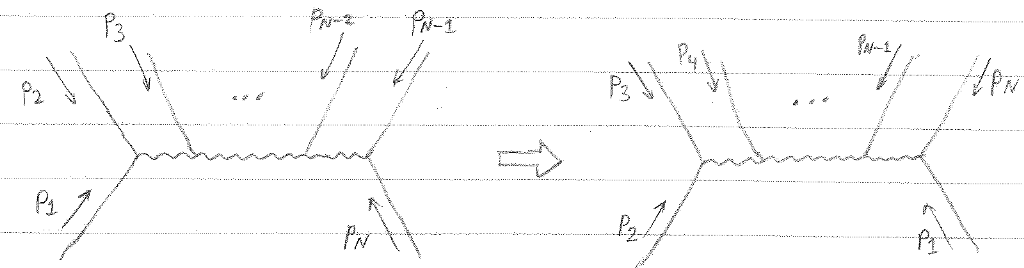
unchanged.

Thus, the amplitude reads:

$$B_N = \int dz_2 \ldots dz_N \, dz_1 \, |z_a - z_b||z_b - z_c||z_c - z_a| \prod_{i=1}^{N} |z_{i+1} - z_i|^{-1+\alpha_0} \quad (-1)^{2\alpha_0}$$

cancel.

$$\langle 0;0| \, \hat{V}(p_1, z_N) \, \hat{V}(p_N, z_{N-1}) \ldots \hat{V}(p_2, z_1) \, |0;0\rangle \quad (-1)^{-2\alpha_0}$$



Cyclic property allows us to go from factorization of the first configuration to factorization in the second configuration.