More on the Fubini–Veneziano–Gervais field operator

$$\hat{Q}^\mu(z) = \hat{Q}^\mu - 2i\alpha' p^\mu \ln z + i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left(-\frac{z^n}{\sqrt{n}}\hat{a}_n^{\mu\dagger} + \frac{z^{-n}}{\sqrt{n}}\hat{a}_n^{\mu}\right)$$

Why call it a "field operator"? (Note, it is Hermitian)

Take $\ln z = i\tau$ or $z = e^{i\tau}$, and observe:

"ϕ"

$$\hat{Q}^\mu(\tau) = \hat{Q}^\mu + 2\alpha' p^\mu \tau + \underbrace{i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left(-\hat{a}_n^{\mu\dagger}e^{in\tau} + \hat{a}_n^{\mu}e^{-in\tau}\right)}_{\text{looks like a field operator}}$$

Then define a field canonically conjugate to $Q^\mu(\tau)$:

"π"

$$\hat{P}^\mu(\tau) = \frac{1}{2\pi\alpha'}\frac{\partial \hat{Q}^\mu}{\partial\tau} = \frac{1}{\pi}p^\mu + \frac{i}{\pi}\frac{1}{\sqrt{2\alpha'}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left[-\hat{a}_n^{\mu\dagger}(in)e^{in\tau} + a_n^{\mu}(-in)e^{-in\tau}\right]$$

To get correct dimensions

$$= \frac{1}{\pi}p^\mu + \frac{1}{\pi\sqrt{2\alpha'}}\sum_{n=1}^{\infty}\sqrt{n}\left(\hat{a}_n^{\mu\dagger}e^{in\tau} + \hat{a}_n^{\mu}e^{-in\tau}\right)$$

Exercise:

Compute the following commutators

$$\left[Q^\mu(\tau),\ \hat{P}^\mu(\tau')\right] = -2i\, g^{\mu\nu}\,\delta(\tau-\tau')$$

[Like an equal τ commutation relation]

$$\left[\hat{Q}^\mu(\tau),\ \hat{Q}^\mu(\tau')\right] = \begin{cases} 2\pi i\,\alpha' g^{\mu\nu}, & \tau\neq\tau' \\ \text{undefined} & \tau=\tau' \end{cases}$$

[ok, because τ and τ' (z and z') never cross in integration]

$$\left[\hat{P}^\mu(\tau),\ \hat{P}^\mu(\tau')\right] = \frac{-i}{\pi\alpha'}\, g^{\mu\nu}\,\delta'(\tau-\tau')$$

[does not commute at $\tau=\tau'$ – very singular]

Useful formulae:

$$\sum_{n=-\infty}^{\infty} e^{\pm in(\tau-\tau')} = 2\pi\,\delta(\tau-\tau')$$

$$\sum_{n=1}^{\infty}\frac{1}{n}e^{\pm in(\tau-\tau')} = -\ln\left(1-e^{\pm in(\tau-\tau')}\right)$$

Solution:

$$[Q^\mu(\tau), P^\mu(\tau')] = \Big[Q^\mu + 2\alpha' P^\mu \tau + i\sqrt{2\alpha'} \sum_n \frac{1}{\sqrt{n}}\left(-a_n^{\mu\dagger} e^{in\tau} + a_n^\mu e^{-in\tau}\right),$$

$$+ \frac{1}{\pi} P^\mu + \frac{1}{\pi\sqrt{2\alpha'}} \sum_{n'} \sqrt{n'}\left(\hat{a}_{n'}^{\nu\dagger} e^{in'\tau'} + a_{n'}^{\nu} e^{-in'\tau'}\right)\Big]$$

$$= \frac{1}{\pi}\underbrace{[Q^\mu, P^\nu]}_{-ig^{\mu\nu}} + \frac{i}{\pi} \sum_{nn'} \frac{\sqrt{n}}{\sqrt{n'}}\Big(-[\hat{a}_n^{\mu\dagger}, \hat{a}_{n'}^{\nu}]\, e^{+in\tau - in'\tau'}$$

$$+ [\hat{a}_n^{\mu}, \hat{a}_{n'}^{\nu\dagger}]\, e^{-in\tau + in'\tau'}\Big)$$

$$= \frac{-i}{\pi}\alpha' g^{\mu\nu} + \frac{i}{\pi}\sum_{n,n'} \frac{\sqrt{n}}{\sqrt{n'}}\Big(-g^{\mu\nu}\delta_{nn'}\, e^{+in\tau - in'\tau'}$$

$$- g^{\mu\nu}\delta_{nn'}\, e^{-in\tau + in'\tau'}\Big)$$

sum over n':

$$= -\frac{i}{\pi} g^{\mu\nu} - \frac{i}{\pi} g^{\mu\nu} \sum_n \left(e^{in(\tau-\tau')} + e^{-in(\tau-\tau')}\right)$$

take $n \to -n$

$$= \frac{-i}{\pi} g^{\mu\nu}\Big[\cancel{1} + \underbrace{\sum_{n=1}^{\infty} e^{in(\tau-\tau')} + \sum_{n=-\infty}^{-1} e^{in(\tau-\tau')}}_{\sum_{n=-\infty}^{\infty} e^{-in(\tau-\tau')} - \cancel{1}}\Big]$$

$\leftarrow$ $n=0$ case removed.

Notice how the $\hat{Q}^\mu$ and $\hat{p}^\mu$ operators act like the $n=0$ term.

$$= \frac{-i}{\pi} g^{\mu\nu} \times 2\pi\,\delta(\tau-\tau')$$

$$= -2i\, g^{\mu\nu}\,\delta(\tau-\tau') \quad ✓$$

$$[\hat{Q}^\mu(\tau), \hat{Q}^\nu(\tau')] = \Big[\hat{Q}^\mu + 2\alpha'\hat{P}^\mu\tau + i\sqrt{2\alpha'}\sum_n \frac{1}{\sqrt{n}}\left(-\hat{a}_n^{\mu\dagger}e^{in\tau} + \hat{a}_n^\mu e^{-in\tau}\right),$$

$$\hat{Q}^\nu + 2\alpha'\hat{P}^\nu\tau + i\sqrt{2\alpha'}\sum_{n'}\frac{1}{\sqrt{n'}}\left(-\hat{a}_{n'}^{\nu\dagger}e^{in'\tau'} + \hat{a}_{n'}^\nu e^{-in'\tau'}\right)\Big]$$

$$= 2\alpha'\tau'[\hat{Q}^\mu, \hat{P}^\nu] + 2\alpha'\tau[\hat{P}^\mu, \hat{Q}^\mu]$$

$$-2\alpha'\sum_{n,n'}\frac{1}{\sqrt{nn'}}\left(-[\hat{a}_n^{\mu\dagger}, \hat{a}_{n'}^\nu]\, e^{in\tau - in'\tau'} - [\hat{a}_n^\mu, \hat{a}_{n'}^{\nu\dagger}]\, e^{-in\tau + in'\tau'}\right)$$

$$= -2i\alpha'\tau' g^{\mu\nu} + 2i\alpha'\tau\, g^{\mu\nu}$$

$$+2\alpha'\sum_{n,n'}\frac{1}{\sqrt{nn'}}\left(g^{\mu\nu}\delta_{nn'}\, e^{in\tau - in'\tau'} - g^{\mu\nu}\delta_{nn'}e^{-in\tau+in'\tau'}\right)$$

sum over n':

$$= 2i\alpha' g^{\mu\nu}(\tau-\tau') + 2\alpha' g^{\mu\nu}\sum_n \frac{1}{n}\left(e^{in(\tau-\tau')} - e^{-in(\tau-\tau')}\right)$$

Perform sum over n:

$$= 2i\alpha' g^{\mu\nu}(\tau-\tau') + 2\alpha' g^{\mu\nu}\left[-\ln\left(1-e^{i(\tau-\tau')}\right) + \ln\left(1-e^{-i(\tau-\tau')}\right)\right]$$

$$[\ldots] = \ln\left(\frac{1-e^{-i(\tau-\tau')}}{1-e^{+i(\tau-\tau')}}\right) = \ln\left(-e^{-i(\tau-\tau')}\right)$$

$$= 2i\alpha' g^{\mu\nu}(\tau-\tau') + 2\alpha' g^{\mu\nu}\left[\ln(-1) + \ln\left(e^{-i(\tau-\tau)}\right)\right]$$

$$= \cancel{2i\alpha' g^{\mu\nu}(\tau-\tau')} + 2\alpha' g^{\mu\nu}(i\pi) - \cancel{2i\alpha' g^{\mu\nu}(\tau-\tau')}$$

$$= 2\pi i\,\alpha' g^{\mu\nu} \quad \checkmark$$

(see Mandelstam, Dual-resonance Models)

$$[P^\mu(\tau), P^\nu(\tau')] = \Big[\frac{1}{\pi} p^\mu + \frac{1}{\pi\sqrt{2\alpha'}} \sum_{n=1}^{\infty} \sqrt{n} \left(\hat{a}_n^{\mu\dagger} e^{in\tau} + \hat{a}_n^{\mu} e^{-in\tau} \right),$$

$$\frac{1}{\pi} p^\nu + \frac{1}{\pi\sqrt{2\alpha'}} \sum_{n'=1}^{\infty} \sqrt{n'} \left(\hat{a}_{n'}^{\nu\dagger} e^{in'\tau'} + \hat{a}_{n'}^{\nu} e^{-in'\tau'} \right) \Big]$$

$$= \frac{1}{2\pi^2\alpha'} \sum_{n,n'} \sqrt{nn'} \Big(\underbrace{[\hat{a}_n^{\mu\dagger}, \hat{a}_{n'}^{\nu}]}_{g^{\mu\nu}\delta_{nn'}} e^{in\tau - in'\tau'} + \underbrace{[a_n^{\mu}, a_{n'}^{\nu\dagger}]}_{-g^{\mu\nu}\delta_{nn'}} e^{-in\tau + in'\tau'} \Big)$$

$$= \frac{1}{2\pi^2\alpha'} g^{\mu\nu} \sum_{n=1}^{\infty} n \Big(e^{in(\tau-\tau')} - \underbrace{e^{-in(\tau-\tau')}}_{\text{take } n \to -n} \Big)$$

$$= \frac{1}{2\pi^2\alpha'} g^{\mu\nu} \Big[\sum_{n=1}^{\infty} n e^{in(\tau-\tau')} + \sum_{n=-\infty}^{-1} n e^{in(\tau-\tau')} \Big]$$

$$= \frac{1}{2\pi^2\alpha'} g^{\mu\nu} \Big[\sum_{n=-\infty}^{\infty} n e^{in(\tau-\tau')} - 0 \Big]$$

n=0 term subtracted.

$$= \frac{1}{2\pi^2\alpha'} g^{\mu\nu} \frac{d}{d(i\tau)} \sum_{n=-\infty}^{\infty} e^{in(\tau-\tau')}$$

$$= \frac{-i}{2\pi^2\alpha'} g^{\mu\nu} \frac{d}{d\tau} 2\pi \delta(\tau-\tau')$$

$$= \frac{-i}{\pi\alpha'} g^{\mu\nu} \delta'(\tau-\tau') \quad \checkmark$$