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Transformation properties of $Q^\mu(\tau)$:

Under projective transformations, $\hat{Q}^\mu(\tau) \to \hat{Q}^\mu(\tau) + \delta\hat{Q}^\mu(\tau)$
(not transforming τ)

$$\delta\hat{Q}^\mu(\tau) \sim [L_n, \hat{Q}^\mu(\tau)]$$

$$\hat{Q}^\mu(\tau) = \hat{Q}^\mu + 2\alpha'\hat{p}^\mu\tau + i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left(-\hat{a}_n^{\mu\dagger}e^{in\tau} + \hat{a}_n^\mu e^{-in\tau}\right)$$

$$\hat{L}_0 = -\alpha'\hat{p}^2 + \sum_{m=1}^{\infty}\left(-m\hat{a}_m^\dagger\cdot a_m\right)$$

So,

$$[\hat{L}_0, \hat{Q}^\mu(\tau)] = [-\alpha'\hat{p}^2, \hat{Q}^\mu] + \left[\sum_m(-m\hat{a}_m^\dagger\cdot\hat{a}_m),\ i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left(-\hat{a}_n^{\mu\dagger}e^{in\tau} + \hat{a}_n^\mu e^{-in\tau}\right)\right]$$

$$= -\alpha'\underbrace{[\hat{p}^2, \hat{Q}^\mu]}_{2i\hat{p}^\mu} - i\sqrt{2\alpha'}\sum_{m,n}\frac{m}{\sqrt{n}}\left(-\underbrace{[\hat{a}_m^\dagger\cdot\hat{a}_m, \hat{a}_n^{\mu\dagger}]}_{-\hat{a}_m^{\mu\dagger}\delta_{m,n}}e^{in\tau} + \underbrace{[\hat{a}_m^\dagger\cdot\hat{a}_m, \hat{a}_n^\mu]}_{\hat{a}_m^\mu\delta_{m,n}}e^{-in\tau}\right)$$

Sum over m, fixing $m \to n$

$$= -2i\alpha'\hat{p}^\mu - i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\sqrt{n}\left(\hat{a}_n^{\mu\dagger}e^{in\tau} + \hat{a}_n^\mu e^{-in\tau}\right)$$

$$\boxed{[\hat{L}_0, \hat{Q}^\mu(\tau)] = -i\frac{\partial\hat{Q}^\mu(\tau)}{\partial\tau}}$$

(!!) Thus L_0, our 'Hamiltonian' generates τ translations!
C.f. Heisenberg equation of motion.

that is, $\hat{Q}^\mu(\tau) = e^{i\hat{L}_0\tau}\hat{Q}(\tau=0)e^{-i\hat{L}_0\tau}$

substitute: $\tau = -i\ln z$

use $\dfrac{\partial}{\partial\tau} = \dfrac{\partial z}{\partial\tau}\dfrac{\partial}{\partial z} = iz\dfrac{\partial}{\partial z}$

$$\Rightarrow [\hat{L}_0, \hat{Q}^\mu(z)] = z\frac{\partial\hat{Q}^\mu(z)}{\partial z}$$

$$\hat{Q}(z) = e^{\hat{L}_0\ln z}\hat{Q}(z=1)e^{-\hat{L}_0\ln z}$$

$$\Rightarrow \quad = z^{\hat{L}_0}\hat{Q}(z=1)z^{-\hat{L}_0}$$

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Next,

$$\hat{L}_1 = \sum_{m=1}^{\infty} \sqrt{m(m+1)}\left(-\hat{a}_m^{\dagger}\cdot\hat{a}_{m+1}\right) - \sqrt{2\alpha'}\,\hat{P}\cdot\hat{a}_1$$

$$[\hat{L}_1, \hat{Q}^\mu(\tau)] = \left[\sum_{m=1}^{\infty}\sqrt{m(m+1)}\left(-\hat{a}_m^\dagger\cdot\hat{a}_{m+1}\right),\ i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left(-\hat{a}_n^{\mu\dagger}e^{in\tau} + \hat{a}_n^{\mu}e^{-in\tau}\right)\right]$$

$$+\left[-\sqrt{2\alpha'}\,\hat{P}\cdot\hat{a}_1,\ \hat{Q}^\mu\right] + \left[-\sqrt{2\alpha'}\,\hat{P}\cdot\hat{a}_1,\ i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left(-\hat{a}_n^{\mu\dagger}e^{in\tau}+\dots\right)\right]$$

$$= -i\sqrt{2\alpha'}\sum_{m,n}\frac{\sqrt{m(m+1)}}{\sqrt{n}}\left(-\overbrace{[\hat{a}_m^\dagger\cdot\hat{a}_{m+1},\ \hat{a}_n^{\mu\dagger}]}^{-a_m^{\mu\dagger}\delta_{m+1,n}}e^{in\tau} + \overbrace{[\hat{a}_m^\dagger\cdot\hat{a}_{m+1},\ \hat{a}_n^\mu]}^{a_{m+1}^\mu\delta_{m,n}}e^{-in\tau}\right)$$

$$-\sqrt{2\alpha'}\,\hat{a}_1^\nu\underbrace{[\hat{P}^\nu, \hat{Q}^\mu]}_{ig^{\mu\nu}} + 2i\alpha'\hat{P}^\nu\sum_n\frac{1}{\sqrt{n}}\underbrace{[\hat{a}_1^\nu, \hat{a}_n^{\mu\dagger}]}_{-g^{\mu\nu}\delta_{1,n}}e^{in\tau}$$

Sum over n in all terms

$$= -i\sqrt{2\alpha'}\sum_{m=1}^{\infty}\left(\frac{\sqrt{m\cancel{(m+1)}}}{\cancel{\sqrt{m+1}}}\hat{a}_m^{\mu\dagger}e^{i(m+1)\tau} + \frac{\sqrt{\cancel{m}(m+1)}}{\cancel{\sqrt{m}}}\hat{a}_{m+1}^{\mu}e^{-im\tau}\right)$$

ch. var: $m \to m-1$

$$-i\sqrt{2\alpha'}\,\hat{a}_1^\mu - 2i\alpha' P^\mu\frac{1}{\sqrt{1}}e^{i\tau}$$

$$= -i\sqrt{2\alpha'}\left(\sum_{m=1}^{\infty}\sqrt{m}\,\hat{a}_m^{\mu\dagger}e^{i(m+1)\tau} + \sum_{m=2}^{\infty}\sqrt{m}\,\hat{a}_m^\mu e^{-i(m-1)\tau}\right) - i\sqrt{2\alpha'}\,a_1^\mu$$

this is the $m=1$ term

$$-2i\alpha'\hat{p}^\mu e^{i\tau}$$

Factor out $e^{i\tau}$ (rename $m \to n$)

$$= -ie^{i\tau}\left[2\alpha'\hat{p}^\mu + \sqrt{2\alpha'}\sum_{n=1}^{\infty}\sqrt{n}\left(\hat{a}_n^{\mu\dagger}e^{in\tau} + \hat{a}_n^\mu e^{-in\tau}\right)\right]$$

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$$\therefore \boxed{[\hat{L}_1, \hat{Q}^\mu(\tau)] = -ie^{i\tau}\frac{\partial \hat{Q}^\mu(\tau)}{\partial \tau} \Rightarrow [\hat{L}_1, \hat{Q}^\mu(z)] = z^2 \frac{\partial Q^\mu(z)}{\partial z}}$$

Take h.c of commutator to get $[\hat{L}_{-1}, \hat{Q}^\mu(\tau)]$:

$$[\hat{L}_1, \hat{Q}^\mu(\tau)]^\dagger = -[\hat{L}_{-1}, \hat{Q}^\mu(\tau)] = ie^{-i\tau}\frac{\partial \hat{Q}^\mu(\tau)}{\partial \tau}$$

$$\therefore \boxed{[\hat{L}_{-1}, \hat{Q}^\mu(\tau)] = -ie^{-i\tau}\frac{\partial \hat{Q}^\mu(\tau)}{\partial \tau} \Rightarrow [\hat{L}_{-1}, \hat{Q}^\mu(z)] = \frac{\partial Q^\mu(z)}{\partial z}}$$

Summary:

$$[\hat{L}_{-1}, \hat{Q}^\mu(z)] = \frac{\partial Q^\mu(z)}{\partial z}$$

$$[\hat{L}_0, \hat{Q}^\mu(z)] = z\frac{\partial Q^\mu(z)}{\partial z}$$

$$[L_1, \hat{Q}^\mu(z)] = z^2\frac{\partial Q^\mu(z)}{\partial z}$$

More generally, $[L_n, \hat{Q}^\mu(z)] = z^n \; z \frac{\partial Q^\mu(z)}{\partial z}$