Conformal generators (Virasoro operators)

$$\boxed{\hat{L}_n = -\alpha'\pi^2 \frac{1}{2\pi}\int_{-\pi}^{\pi} d\tau\, e^{i\tau n} :\hat{P}^2(\tau):}$$

Virasoro generator

$$\hat{P}^2(\tau) = \left[\frac{1}{\pi}\hat{p}^{\mu} + \frac{1}{\pi\sqrt{2\alpha'}}\sum_m \sqrt{m}\left(\hat{a}_m^{\dagger} e^{im\tau} + \hat{a}_m e^{-im\tau}\right)\right]^2$$

$$\Rightarrow \; :\hat{P}^2(\tau): \; = \frac{1}{\pi}\hat{p}^2 + \frac{1}{2\alpha'\pi^2}\sum_{mm'}\sqrt{mm'}\Big(\hat{a}_m^{\dagger}\cdot\hat{a}_{m'}^{\dagger} e^{i(m+m')} + \hat{a}_m^{\dagger}\cdot\hat{a}_{m'} e^{i(m-m')\tau}$$
$$+ \hat{a}_{m'}^{\dagger}\cdot\hat{a}_m e^{-i(m-m')\tau} + \hat{a}_m\cdot\hat{a}_{m'} e^{-i(m+m')\tau}\Big)$$

cross term

$$+ \frac{2}{\pi^2\sqrt{2\alpha'}}\hat{p}\cdot\sum_m \sqrt{m}\left(\hat{a}_m^{\dagger} e^{im\tau} + \hat{a}_m e^{-im\tau}\right)$$

Plug into $\hat{L}_n = \ldots\int_{-\pi}^{\pi} d\tau\, e^{i\tau n} :\hat{P}^2(\tau):$

$$\hat{L}_n = -\alpha'\pi^2\frac{1}{2\pi}\int_{-\pi}^{\pi} d\tau\Big[\frac{1}{\pi^2}\hat{p}^2 e^{i\tau n} + \frac{1}{2\alpha'\pi^2}\sum_{m,m'}\sqrt{mm'}\Big(\hat{a}_m^{\dagger}\cdot\hat{a}_{m'} e^{i(m+m'+n)\tau}$$
$$+ \hat{a}_m^{\dagger}\cdot\hat{a}_{m'} e^{i(m-m'+n)\tau} + \hat{a}_{m'}^{\dagger}\cdot\hat{a}_m e^{-i(m-m'-n)\tau} + \hat{a}_m\cdot\hat{a}_{m'} e^{-i(m+m'-n)\tau}\Big)$$
$$+ \frac{1}{\pi^2}\sqrt{\frac{2}{\alpha'}}\hat{p}\cdot\sum_m\sqrt{m}\left(\hat{a}_m^{\dagger} e^{i(m+n)\tau} + \hat{a}_m e^{-i(m-n)\tau}\right)\Big]$$

Use $\frac{1}{2\pi}\int_{-\pi}^{\pi} d\tau\, e^{i\tau n} = \frac{2\sin(\pi n)}{2\pi n} = \delta_{0,n} = \delta_n$ (shorthand) $\quad n\in\mathbb{Z}$

$$\hat{L}_n = -\alpha'\pi^2\Big[\frac{1}{\pi^2}\hat{p}^2\delta_n + \frac{1}{2\alpha'\pi^2}\sum_{m,m'}\sqrt{mm'}\Big(\hat{a}_m^{\dagger}\cdot\hat{a}_{m'}^{\dagger}\delta_{m+m'+n} + \hat{a}_m^{\dagger}\cdot\hat{a}_{m'}\delta_{m-m'+n}$$
$$+ \hat{a}_{m'}^{\dagger}\cdot\hat{a}_m\delta_{m-m'-n} + \hat{a}_m\cdot\hat{a}_{m'}\delta_{m+m'+n}\Big)$$
$$+ \frac{1}{\pi^2}\sqrt{\frac{2}{\alpha'}}\hat{p}\cdot\sum_m\sqrt{m}\left(\hat{a}_m^{\dagger}\delta_{m+n} + \hat{a}_m\delta_{m-n}\right)\Big]$$

Sum over m'

$$\hat{L}_n = -\alpha' \hat{p}^2 \delta_n + \frac{1}{2}\sum_{m=1}^{\infty}\Big[\sqrt{m(-m-n)}\left(-\hat{a}_m^\dagger\cdot\hat{a}^\dagger_{-m-n}\right) + \sqrt{m(m+n)}\left(-\hat{a}_m^\dagger\cdot\hat{a}_{m+n}\right)$$

$$+ \sqrt{m(m-n)}\left(-\hat{a}^\dagger_{m-n}\cdot\hat{a}_m\right) + \sqrt{m(-m+n)}\left(-\hat{a}_m\cdot\hat{a}_{-m+n}\right)\Big]$$

$$-\sqrt{2\alpha'}\,\hat{P}\cdot\left(\sqrt{-n}\,\hat{a}^\dagger_{-n} + \sqrt{n}\,\hat{a}_n\right)$$

Note: for $n \le 0$, $\hat{a}_n^\dagger = \hat{a}_n = 0$ (they do not exist)

To obtain the canonical forms, take cases: $n=0$, $n>0$, $n<0$.

$$\boxed{\hat{L}_0 = -\alpha'\hat{p}^2 + \sum_{m=1}^{\infty}\left(-m\hat{a}_m^\dagger\cdot\hat{a}_m\right)}$$

take $m \to m+n$

$$\hat{L}_n = \frac{1}{2}\sum_{m=1}^{\infty}\sqrt{m(m+n)}\left(-\hat{a}_m^\dagger\cdot\hat{a}_{m+n}\right) + \frac{1}{2}\sum_{m=n+1}^{\infty}\sqrt{m(m-n)}\left(-\hat{a}^\dagger_{m-n}\cdot\hat{a}_m\right)$$

$$+\frac{1}{2}\sum_{m=1}^{n-1}\sqrt{m(-m+n)}\;\hat{a}_m\cdot\hat{a}_{-m+n} - \sqrt{2\alpha' n}\;\hat{P}\cdot\hat{a}_n$$

Both $n>0$

$$\boxed{\hat{L}_n = \sum_{m=1}^{\infty}\sqrt{m(m+n)}\left(-\hat{a}_m^\dagger\cdot\hat{a}_{m+n}\right) + \frac{1}{2}\sum_{m=1}^{n-1}\sqrt{m(-m+n)}\left(-\hat{a}_m\cdot\hat{a}_{-m+n}\right) - \sqrt{2\alpha' n}\;\hat{P}\cdot\hat{a}_n}$$

Similarly,

$$\boxed{\hat{L}_{-n} = \sum_{m=1}^{\infty}\sqrt{m(m+n)}\left(-\hat{a}^\dagger_{m+n}\cdot\hat{a}_m\right) + \frac{1}{2}\sum_{m=1}^{n-1}\sqrt{m(-m+n)}\left(-\hat{a}_m^\dagger\cdot\hat{a}^\dagger_{-m+n}\right) - \sqrt{2n\alpha'}\;\hat{P}\cdot\hat{a}_n^\dagger}$$

$$\equiv \hat{L}_n^\dagger$$

Notice: $\hat{L}_n^\dagger$ creates oscillations, $\hat{L}_n$ annihilates oscillations

Virasoro algebra:

space-time dimension

$$[\hat{L}_m, \hat{L}_n] = (m-n)\hat{L}_{m+n} + \underbrace{\frac{d}{12}m(m^2-1)\,\delta_{m+n,0}}_{\text{"anomaly term"}}$$

"anomaly term" – arises from commutators of the form $[a^\dagger\cdot a^\dagger, a\cdot a]$, and by normal ordering.