Verifying Virasoro algebra

Identities:

$$[a_A^\dagger\cdot a_B,\ a_C^\dagger\cdot a_D] = a_C^\dagger\cdot a_B\ \delta_{AD} - a_A^\dagger\cdot a_D\ \delta_{BC}$$

$$[a_A^\dagger\cdot a_B,\ a_C\cdot a_D] = a_B\cdot a_D\ \delta_{AC} + a_B\cdot a_C\ \delta_{AD}$$

$$[a_A^\dagger\cdot a_B,\ a_C^\mu] = a_B^\mu\ \delta_{AC}$$

$$[a_A^\dagger\cdot a_B^\dagger,\ a_C\cdot a_D] = a_A^\dagger\cdot a_D\ \delta_{BC} + a_B^\dagger\cdot a_D\ \delta_{AC} + (a_A^\dagger\cdot a_C + \overset{=d\ (\text{space time dimensions})}{g^\mu{}_\mu}\ \delta_{AC})\delta_{BD}$$
$$+ (a_B^\dagger\cdot a_C + g^\mu{}_\mu\ \delta_{BC})\delta_{AD}$$

(others obtainable by taking h.c: $[A,B]^\dagger = -[A^\dagger, B^\dagger]$)

Case 1: $m, n > 0$

$$[L_m, L_n] = \sum_{r,r'} \sqrt{r(r+m)}\sqrt{r'(r'+n)}\ \underbrace{[a_r^\dagger\cdot a_{r+m},\ a_{r'}^\dagger\cdot a_{r'+n}]}_{a_r^\dagger\cdot a_{r+m}\delta_{r,r'+n} - a_{r'}^\dagger\cdot a_{r'+n}\delta_{r+m,r'}} \qquad \text{Sum over } r'$$

$$+ \sum_{r=1}^{\infty}\sum_{r'=1}^{n-1} \frac{1}{2}\sqrt{r(r+m)}\sqrt{r'(-r'+n)}\ \underbrace{[a_r^\dagger\cdot a_{r+m},\ a_{r'}\cdot a_{-r'+n}]}_{a_{r+m}\cdot a_{-r'+n}\delta_{r,r'} + a_{r+m}\cdot a_{r'}\delta_{r,-r'+n}} \qquad \text{Sum over } r$$

$$+ \sum_{r=1}^{\infty} \sqrt{2\alpha'}\sqrt{r(r+m)}\sqrt{n}\ \underbrace{[a_r^\dagger\cdot a_{r+m},\ P.a_n]}_{P.a_{r+m}\delta_{r,n}} \qquad \text{Sum over } r$$

$$+ \sum_{r'=1}^{m-1}\sum_{r=1}^{\infty} \frac{1}{2}\sqrt{r'(-r'+m)}\sqrt{r(r+n)}\ [a_{r'}\cdot a_{r'+m},\ a_r^\dagger\cdot a_{r+n}]$$

rename $r\leftrightarrow r'$: Same as $(-1)\times$ 2nd term with $m\leftrightarrow n$

$$+ \sum_{r=1}^{\infty} \sqrt{2\alpha'}\sqrt{m}\sqrt{r(r+n)}\ [P.a_m,\ a_r^\dagger\cdot a_{r+n}]$$

rename $r\leftrightarrow r'$ Same as $(-1)\times$ 3rd term with $m\leftrightarrow n$.

Put 2nd and 4th terms together
" 3rd and 5th " "

$$[L_m, L_n] = \sum_{r=n+1}^{\infty} \sqrt{r(r+m)}\sqrt{(r-n)r}\; a^\dagger_{r-n}\cdot a_{r+m} + \sum_{r=1}^{\infty} \sqrt{r(r+m)}\sqrt{(r+m)(r+m+n)}\left(-a^\dagger_r\cdot a_{r+m+n}\right)$$

$$+ \sum_{r'=1}^{n-1} \frac{1}{2}\left[\sqrt{r'(r'+m)}\sqrt{r'(-r'+n)}\; a_{r'+m}\, a_{-r'+n} + \sqrt{(-r'+n)(-r'+m+n)}\sqrt{r'(-r'+n)}\; a_{-r'+m+n}\cdot a_{r'}\right]$$

$$- \sum_{r'=1}^{m-1} \frac{1}{2}\left[\sqrt{r'(r'+n)}\sqrt{r'(-r'+m)}\; a_{r'+n}\, a_{-r'+m} + \sqrt{(-r'+m)(-r'+m+n)}\sqrt{r'(-r'+m)}\; a_{-r'+m+n}\cdot a_{r'}\right]$$

$$+ \sqrt{2\alpha'}\left(\sqrt{n(m+n)}\sqrt{n} - \sqrt{m(m+n)}\sqrt{m}\right) P.\, a_{m+n}$$

take $r \to r+n$

$$= \sum_{r=n+1}^{\infty} r\sqrt{(r+m)(r-n)}\; a^\dagger_{r-n}\cdot a_{r+m} + \sum_{r=1}^{\infty} (r+m)\sqrt{r(r+m+n)}\left(-a^\dagger_r\cdot a_{r+m+n}\right)$$

Take $r' \to r'-m$

$$+ \sum_{r'=1}^{n-1} \frac{1}{2}\left[r'\sqrt{(r'+m)(-r'+n)}\; a_{r'+m}a_{-r'+n} + (-r'+n)\sqrt{(-r'+m+n)r'}\; a_{-r'+m+n}\cdot a_{r'}\right]$$

$$+ \sum_{r'=1}^{m-1} \frac{1}{2}\left[-r'\sqrt{(r'+n)(-r'+m)}\; a_{r'+n}a_{-r'+m} - (-r'+m)\sqrt{(-r'+m+n)r'}\; a_{-r'+m+n}\cdot a_{r'}\right.$$

Take $r' \to r'-n$

$$+ \sqrt{2\alpha'}\left(n\sqrt{m+n} - m\sqrt{m+n}\right) P.\, a_{m+n}$$

$$= \sum_{r=1}^{\infty} -(r+n)\sqrt{(r+m+n)r}\left(-a^\dagger_r\cdot a_{r+m+n}\right) + \sum_{r=1}^{\infty} (r+m)\sqrt{r(r+m+n)}\left(-a^\dagger_r\cdot a_{r+m+n}\right)$$

$$+ \sum_{r'=m+1}^{m+n-1} -\frac{1}{2}(r'-m)\sqrt{r'(-r'+m+n)}\left(-a_{r'}\cdot a_{-r'+m+n}\right) + \sum_{r'=1}^{n-1} \frac{-1}{2}(-r'+n)\sqrt{(-r'+m+n)r'}\left(-a_{-r'+m+n}\cdot a_{r'}\right)$$

$$+ \sum_{r'=n+1}^{m+n-1} \frac{1}{2}(r'-n)\sqrt{r'(-r'+m+n)}\; a_{r'}\cdot a_{-r'+m+n} + \sum_{r'=1}^{m-1} \frac{1}{2}(-r'+m)\sqrt{(-r'+m+n)r'}\left(-a_{-r'+m+n}\cdot a_{r'}\right)$$

$$- (m-n)\sqrt{2\alpha'(m+n)}\; P.\, a_{m+n}$$

Combine 1^{st} & 2^{nd} terms

combine 3^{rd} & 6^{th} terms: $r' = m$ term makes no contribution
combine 4^{th} & 5^{th} terms: $r' = n$ term makes no contribution.

$\Big\}$ drop primes on r'.

$$= \sum_{r=1}^{\infty} \left[-(\cancel{r}+n) + (\cancel{r}+m) \right] \sqrt{r(r+m+n)} \left(-a_r^{\dagger} \cdot a_{r+m+n} \right)$$

$$+ \sum_{r=1}^{m+n-1} \frac{1}{2} (-r+m) \sqrt{r(-r+m+n)} \left(-a_r \cdot a_{-r+m+n} \right)$$

$$+ \sum_{r=1}^{m+n-1} \frac{1}{2} (r-n) \sqrt{r(-r+m+n)} \left(-a_r \cdot a_{-r+m+n} \right)$$

$\Big\}$ combine.

$$-(m-n)\sqrt{2\alpha'(m+n)}\; P \cdot a_{m+n}$$

$$= (m-n) \left\{ \sum_{r=1}^{\infty} \sqrt{r(r+m+n)} \left(-a_r^{\dagger} \cdot a_{r+m+n} \right) + \frac{1}{2} \sum_{r=1}^{m+n-1} \sqrt{r(-r+m+n)} \left(-a_r \cdot a_{-r+m+n} \right) - \sqrt{2\alpha'(m+n)}\; P \cdot a_{m+n} \right\}$$

$$= (m-n) \hat{L}_{m+n}$$

To get the anomaly term, must consider $n>0$, $m<0$ case.