On-shell conditions

Recall operator expression for scattering amplitude in multiperipheral configuration:

$$B_N = \langle 0; p_N | \hat{V}(p_{N-1}) \hat{D}(s_{1,N-1}) \cdots \hat{V}(p_4) \hat{D}(s_{13}) \hat{V}(p_3) \hat{D}(s_{12}) \hat{V}(p_2) | 0; p_1 \rangle$$

To each vertex, I've included $e^{-ip\cdot Q}$ for later convenience.

To factorize, we wrote one of the propagators as:

$$\hat{D}(s) = \int_0^1 dy\, (1-y)^{-1+\alpha_0}\, y^{-1-\alpha(s)}\, y^{\sum_n (-n a_n^\dagger \cdot a_n)}$$

$$= \sum_{r_0=0}^{\infty} \binom{r_0-\alpha_0}{r_0} \frac{1}{-\alpha(s) + \sum_n (-n a_n^\dagger \cdot a_n) + r_0}$$

Write $\alpha(s) = \alpha' s + \alpha_0$
$= \alpha' \hat{p}^2 + \alpha_0$

will give $(p_1 + \dots + p_i)^2 = s_{1i}$ ✓

Can write in terms of $L_0 = -\alpha' \hat{p}^2 + \sum_n (-n a_n^\dagger a_n)$

$$\sum_n (-n a_n^\dagger \cdot a_n) = L_0 + \alpha' \hat{p}^2$$

$$\hat{D}(s) = \sum_{r_0=0}^{\infty} \binom{r_0-\alpha_0}{r_0} \frac{1}{-\alpha' \hat{p}^2 - \alpha_0 + \hat{L}_0 + \alpha' \hat{p}^2 + r_0}$$

$$= \sum_{r_0=0}^{\infty} \binom{r_0-\alpha_0}{r_0} \frac{1}{\hat{L}_0 - \alpha_0 + r_0} \equiv \hat{D} \qquad \left(\text{no longer explicitly dependent on } s\right)$$

so that

$$B_N = \langle 0; p_N | \hat{V} \hat{D} \hat{V} \cdots \sum_{r_0} \binom{r_0-\alpha_0}{r_0} \frac{1}{\hat{L}_0 - \alpha_0 - r_0} \hat{V} \hat{D} \hat{V} \cdots | 0; p_1 \rangle$$

Now, we will get a pole (correspondingly, a particle) when this $\hat{L}_0$ gives an integer equal to $\alpha_0 - r_0$ upon acting on everything to the right (or left):

<u>pole condition</u>: $$\hat{L}_0 \underbrace{\left[\hat{V} \hat{D} \hat{V} \cdots | 0; p_1 \rangle \right]}_{\text{"tree"}} = (\alpha_0 - r_0) \underbrace{\left[\hat{V} \hat{D} \hat{V} \cdots | 0; p_1 \rangle \right]}_{\text{"tree"}}$$

Since $[\hat{L}_0, \hat{D}] = 0$, it proves convenient to define $|\text{tree}\rangle$ with an extra factor of $\hat{D}$ at end:

$$|\text{tree}\rangle = \hat{D}\,\hat{V}(p_j)\,\hat{D}\,\hat{V}(p_{j-1}) \cdots |0; p_1\rangle$$

On-shell condition:

$$\hat{L}_0 |\text{tree}\rangle = (\alpha_0 - r_0) |\text{tree}\rangle$$

or $$\boxed{(\hat{L}_0 - \alpha_0 + r_0) |\text{tree}\rangle = 0}$$

Furthermore, since $\hat{L}_n \sim \sum_m \hat{a}_m \hat{a}_{-m+n}$ for $n > 0$, $\hat{L}_n$ annihilates the physical ground state:

$$\boxed{\hat{L}_n |0; p\rangle = 0 \quad \text{for } n > 0}$$

On shell gauge conditions

For unit intercept, $\alpha_0 = 1$, there exists a remarkable set of on-shell gauge conditions satisfied by tree amplitudes (discovered by Virasoro):

First, the on-shell condition becomes ($\alpha_0 = 1 \Rightarrow r_0 = 0$)

$$(\hat{L}_0 - 1)\,|\text{tree}\rangle = 0.$$

Now derive two identities:

①
$$\begin{aligned}
(\hat{L}_0 + n - 1 - \hat{L}_n)\,\hat{V}(p) &= \hat{V}(p)(\hat{L}_0 + n - 1 - \hat{L}_n) + [\hat{L}_0, \hat{V}(p)] - [\hat{L}_n, \hat{V}(p)] \\
&= \hat{V}(p)(\hat{L}_0 + n - 1 - \hat{L}_n) \\
z = 1 \longrightarrow \quad &\quad + z^0\left(z\frac{\partial}{\partial z} + (0+1)\alpha_0\right)\hat{V}(p) - z^n\left(z\frac{\partial}{\partial z} + (n+1)\alpha_0\right)\hat{V}(p) \\
&= \hat{V}(p)(\hat{L}_0 + n - 1 - \hat{L}_n) - n\alpha_0\,\hat{V}(p). \\
&= \hat{V}(p)(\hat{L}_0 - 1 - \hat{L}_n)
\end{aligned}$$

$\alpha_0 = 1$ is crucial!

$\hat{V}(p)$ must have unit conformal weight.

$\boxed{\Rightarrow \text{Removes an } n}$

②
$$\begin{aligned}
(\hat{L}_0 - 1 - \hat{L}_n)\underbrace{\frac{1}{\hat{L}_0 - 1}} &= 1 - \hat{L}_n \frac{1}{\hat{L}_0 - 1} \\
&= 1 + \hat{L}_n \sum_{m=0}^{\infty} \hat{L}_0^{\,m} \\
&= 1 + \sum_{m=0}^{\infty} (\hat{L}_0 + n)^m \hat{L}_n \\
&= 1 - \frac{1}{\hat{L}_0 + n - 1}\hat{L}_n \\
&= \frac{1}{\hat{L}_0 + n - 1}(\hat{L}_0 + n - 1 - \hat{L}_n) \quad ✓
\end{aligned}$$

for $\alpha_0 = 1$, this is D.

$\boxed{\Rightarrow \text{restores an } n}$

Now consider $(L_0 - 1 - \hat{L}_n)\,|\text{tree}\rangle = (L_0 - 1 - \hat{L}_n)\,\hat{D}\,\hat{V}\hat{D}\ldots|0;p_1\rangle$

$$(L_0 - 1 - \hat{L}_n)\,|\text{tree}\rangle = (L_0 - 1 - \hat{L}_n)\,\hat{D}\hat{V}\hat{D}\hat{V}\cdots|0;p_1\rangle$$

Note: $\hat{D} = \sum_{r_0=0}^{\infty}\binom{r_0-\alpha_0}{r_0}\frac{1}{\hat{L}_0-\alpha_0+r_0} = \frac{1}{\hat{L}_0-1}$ for $\alpha_0 = 1$

Now push $(L_0 - 1 - \hat{L}_n)$ through train of $\hat{D}\hat{V}(s)$ using the identities ① & ②:

$$\underbrace{(\hat{L}_0 - 1}_{\text{vanishes by on-shell condition}} - \hat{L}_n)\,|\text{tree}\rangle = \frac{1}{\hat{L}_0+n-1}\hat{V}\frac{1}{\hat{L}_0+n-1}\hat{V}\cdots\underbrace{(L_0 - 1 - \hat{L}_n)}_{\text{But these vanish by the on-shell condition.}}|0;p_1\rangle = 0$$

$$\boxed{\Rightarrow \quad \hat{L}_n\,|\text{tree}\rangle = 0, \quad \text{for } n>0}$$

These are the subsidiary on-shell "gauge" conditions one gets when setting $\alpha_0 = 1$.

So, in summary, for $\alpha_0 = 1$ we have for physical states:

$(L_0 - 1)\,|\text{tree}\rangle = 0$ "physical" on-shell condition

$L_n\,|\text{tree}\rangle = 0$ on-shell "gauge" condition ($n>0$)