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Regge behavior of Virasoro amplitude

Asymptotic expansion:

$$\Gamma(z) \xrightarrow[|z|\to\infty]{} e^{-z} z^{z}\sqrt{2\pi z}\,\left(1+\mathcal{O}\left(\tfrac{1}{z}\right)\right) \qquad -\pi < \arg(z) < \pi$$

(not on negative real axis)

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \xrightarrow[|z|\to\infty]{} \frac{e^{-z-a}(z+a)^{z+a}\sqrt{2\pi(z+a)}}{e^{-z-b}(z+b)^{z+b}\sqrt{2\pi(z+b)}} = z^{a-b}\left(1+\mathcal{O}\left(\tfrac{1}{z}\right)\right)$$

Also use $\displaystyle \Gamma(z) = \frac{\pi}{\Gamma(1-z)\sin\pi z}$

Start by rewriting the Virasoro amplitude

$$V(s,t,u) = \sqrt{\pi}\,g\,\frac{\Gamma\left(-\frac{\alpha_s}{2}\right)\Gamma\left(-\frac{\alpha_t}{2}\right)\Gamma\left(-\frac{\alpha_u}{2}\right)}{\Gamma\left(-\frac{\alpha_t}{2}-\frac{\alpha_u}{2}\right)\Gamma\left(-\frac{\alpha_s}{2}-\frac{\alpha_u}{2}\right)\Gamma\left(-\frac{\alpha_s}{2}-\frac{\alpha_t}{2}\right)}$$

$$-\alpha_u = \alpha_s+\alpha_t-\sum\alpha, \qquad \text{where} \quad \sum\alpha = 4\alpha' m^2 + 3\alpha_0 = \text{const}$$

$$V(s,t,u) = \sqrt{\pi}\,g\,\frac{\Gamma\left(-\frac{\alpha_s}{2}\right)\Gamma\left(-\frac{\alpha_t}{2}\right)\Gamma\left(\frac{\alpha_s}{2}+\frac{\alpha_t}{2}-\frac{\sum\alpha}{2}\right)}{\Gamma\left(\frac{\alpha_s}{2}-\frac{\sum\alpha}{2}\right)\Gamma\left(\frac{\alpha_t}{2}-\frac{\sum\alpha}{2}\right)\Gamma\left(-\frac{\alpha_s}{2}-\frac{\alpha_t}{2}\right)}$$

then use:

$$\Gamma\left(-\frac{\alpha_s}{2}\right) = \frac{\pi}{\sin\left(-\frac{\pi\alpha_s}{2}\right)\Gamma\left(1+\frac{\alpha_s}{2}\right)} \quad \text{and} \quad \Gamma\left(-\frac{\alpha_s}{2}-\frac{\alpha_t}{2}\right) = \frac{\pi}{\sin\left(-\frac{\pi\alpha_s}{2}-\frac{\pi\alpha_t}{2}\right)\Gamma\left(1+\frac{\alpha_s}{2}+\frac{\alpha_t}{2}\right)}$$

$$V(s,t,u) = \sqrt{\pi}\,g\,\frac{\pi}{\sin\left(-\frac{\pi\alpha_s}{2}\right)\Gamma\left(1+\frac{\alpha_s}{2}\right)}\,\frac{\Gamma\left(-\frac{\alpha_t}{2}\right)\Gamma\left(\frac{\alpha_s}{2}+\frac{\alpha_t}{2}-\frac{\sum\alpha}{2}\right)}{\Gamma\left(\frac{\alpha_s}{2}-\frac{\sum\alpha}{2}\right)\Gamma\left(\frac{\alpha_t}{2}-\frac{\sum\alpha}{2}\right)}\,\frac{\sin\left(-\frac{\pi\alpha_s}{2}-\frac{\pi\alpha_t}{2}\right)\Gamma\left(1+\frac{\alpha_s}{2}+\frac{\alpha_t}{2}\right)}{\pi}$$

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rearrange:

$$V(s,t,u) = \sqrt{\pi}\, g \, \frac{\Gamma\left(\frac{-\alpha_t}{2}\right)}{\Gamma\left(\frac{\alpha_t}{2} - \frac{\sum\alpha}{2}\right)} \left[\frac{\sin\left(\frac{-\pi\alpha_s}{2} - \frac{\pi\alpha_t}{2}\right)}{\sin\left(\frac{-\pi\alpha_s}{2}\right)} \; \frac{\Gamma\left(1 + \frac{\alpha_s}{2} + \frac{\alpha_t}{2}\right)}{\Gamma\left(1 + \frac{\alpha_s}{2}\right)} \; \frac{\Gamma\left(\frac{\alpha_s}{2} + \frac{\alpha_t}{2} - \frac{\sum\alpha}{2}\right)}{\Gamma\left(\frac{\alpha_s}{2} - \frac{\sum\alpha}{2}\right)} \right]$$

Proceed to take Regge limit: Give $\alpha(s)$ small imag. part: $\alpha(s) \equiv (\alpha' + i\epsilon)s + \alpha_0$

start with $\alpha(s) \to \infty$.

$$\frac{\sin\left(\frac{+\pi\alpha_s}{2} + \frac{\pi\alpha_t}{2}\right)}{\sin\left(\frac{+\pi\alpha_s}{2}\right)} = \frac{e^{-\frac{i\pi}{2}(\alpha_s+\alpha_t)} - e^{\frac{i\pi}{2}(\alpha_s+\alpha_t)}}{e^{-\frac{i\pi}{2}\alpha_s} - e^{\frac{i\pi}{2}\alpha_s}} \xrightarrow[\alpha_s \to \infty]{} \frac{e^{-\frac{i\pi}{2}(\alpha_s+\alpha_t)}}{e^{-\frac{i\pi}{2}\alpha_s}} = e^{-\frac{i\pi}{2}\alpha_t}$$

$$\frac{\Gamma\left(1 + \frac{\alpha_s}{2} + \frac{\alpha_t}{2}\right)}{\Gamma\left(1 + \frac{\alpha_s}{2}\right)} \xrightarrow[\alpha_s \to \infty]{} \left(\frac{\alpha_s}{2}\right)^{\alpha_t/2} \left(1 + O\left(\frac{1}{\alpha_s/2}\right)\right)$$

$$\frac{\Gamma\left(\frac{\alpha_s}{2} + \frac{\alpha_t}{2} - \frac{\sum\alpha}{2}\right)}{\Gamma\left(\frac{\alpha_s}{2} - \frac{\sum\alpha}{2}\right)} \xrightarrow[\alpha_s \to \infty]{} \left(\frac{\alpha_s}{2}\right)^{\alpha_t/2} \left(1 + O\left(\frac{1}{\alpha_s/2}\right)\right)$$

So, finally the Regge limit of the Virasoro amplitude is:

$$V(s,t,u) \longrightarrow \sqrt{\pi}\, g \, \frac{\Gamma\left(\frac{-\alpha_t}{2}\right)}{\Gamma\left(\frac{\alpha_t}{2} - \frac{\sum\alpha}{2}\right)} \, e^{-\frac{i\pi}{2}\alpha_t} \left(\frac{\alpha(s)}{2}\right)^{\alpha(t)}$$