Regge behavior of Veneziano amplitude

Start with

$$B_4(s,t) = \frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))},$$

and use asymptotic expansion

$$\Gamma(z) \xrightarrow[|z|\to\infty]{} e^{-z} z^{z} \sqrt{2\pi z}\left(1 + \mathcal{O}\left(\tfrac{1}{z}\right)\right) \qquad -\pi < \arg(z) < \pi$$

(not on negative real axis)

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \xrightarrow[|z|\to\infty]{} \frac{e^{-z-a}(z+a)^{z+a}\sqrt{2\pi(z+a)}}{e^{-z-b}(z+b)^{z+b}\sqrt{2\pi(z+b)}} = z^{a-b}\left(1+\mathcal{O}\left(\tfrac{1}{z}\right)\right)$$

small; drop.

Write using $\Gamma(z) = \dfrac{\pi}{\Gamma(1-z)\sin\pi z}$

$$B_4(s,t) = \left\{\frac{\pi}{\Gamma(1+\alpha(s))\sin(-\pi\alpha(s))}\,\frac{\pi}{\Gamma(1+\alpha(t))\sin(-\pi\alpha(t))}\,\frac{\Gamma(1+\alpha(s)+\alpha(t))\sin(-\pi\alpha(s)-\pi\alpha(t))}{\pi}\right.$$

$$= -\pi\left\{\frac{1}{\Gamma(1+\alpha(t))}\,\frac{\Gamma(1+\alpha(s)+\alpha(t))}{\Gamma(1+\alpha(s))}\,\frac{\sin(\pi\alpha(s)+\pi\alpha(t))}{\sin\pi\alpha(s)\,\sin\pi\alpha(t)}\right\}$$

$$B_4(s,u) = \frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(u))}{\Gamma(-\alpha(s)-\alpha(u))}$$

$$= \frac{\pi}{\Gamma(1+\alpha(s))\sin(-\pi\alpha(s))}\,\frac{\Gamma(-\alpha(u))}{\Gamma(-\alpha(s)-\alpha(u))}$$

$$= -\pi\,\frac{1}{\Gamma(1+\alpha(s))\sin\pi\alpha(s)}\,\frac{\Gamma(\alpha(s)+\alpha(t)-\Sigma\alpha)}{\Gamma(\alpha(t)-\Sigma\alpha)}$$

$-\alpha(u) = \alpha(s)+\alpha(t)-\sum\alpha$

where

$\sum\alpha = 4\alpha' m + 3\alpha_0$

$$B_4(t,u) = \frac{\Gamma(-\alpha(t))\,\Gamma(-\alpha(u))}{\Gamma(-\alpha(t)-\alpha(u))}$$

$$= \frac{-\pi}{\Gamma(1+\alpha(t))\sin\pi\alpha(t)}\,\frac{\Gamma(-\alpha(u))}{\Gamma(-\alpha(t)-\alpha(u))}$$

So, in total

$$V(s,t,u) = -\pi g \left\{ \frac{1}{\Gamma(1+\alpha(t))\sin\pi\alpha(t)} \left(\frac{\Gamma(1+\alpha(s)+\alpha(t))}{\Gamma(1+\alpha(s))} \frac{\sin(\pi\alpha(s)+\pi\alpha(t))}{\sin\pi\alpha(s)} + \frac{\Gamma(-\alpha(u))}{\Gamma(-\alpha(t)-\alpha(u))} \right) + \frac{1}{\Gamma(1+\alpha(s))\sin\pi\alpha(s)} \frac{\Gamma(\alpha(s)+\alpha(t)-\sum\alpha)}{\Gamma(\alpha(t)-\sum\alpha)} \right\}$$

Take $s\to\infty$, fixed t.

If $\alpha(s)$ is taken to be real, we will run into a series of poles from $\frac{1}{\sin\pi\alpha(s)}$.

$\Rightarrow$ Give $\alpha(s)$ a small imaginary part: $\alpha(s) = \alpha_0 + (\alpha' + i\epsilon)s$.

In the third term, $\sin\pi\alpha(s) = \frac{i}{2}\left(e^{-i\pi\alpha(s)} - e^{+i\pi\alpha(s)}\right) \xrightarrow[s\to\infty]{} \frac{i}{2} e^{+\pi\epsilon s}$ is in denom. ~~⇒ vanishes~~

Then $\frac{\Gamma(1+\alpha(s)+\alpha(t))}{\Gamma(1+\alpha(s))} \xrightarrow[s\to\infty]{} \alpha(s)^{\alpha(t)}$

and $$\frac{\Gamma(-\alpha(u))}{\Gamma(-\alpha(t)-\alpha(u))} = \frac{\Gamma(-\alpha_0 - (\alpha'+i\epsilon)u)}{\Gamma(-\alpha(t) - \alpha_0 - (\alpha'+i\epsilon)u)}$$

$$= \frac{\Gamma(-\alpha_0 - (\alpha'+i\epsilon)(-s-t+\sum m^2))}{\Gamma(-\alpha(t) - \alpha_0 - (\alpha'+i\epsilon)(-s-t+\sum m^2))}$$

$$\xrightarrow[s\to\infty]{} (\alpha' s)^{0 - -\alpha(t)} \approx \alpha(s)^{\alpha(t)}$$

and $$\frac{\sin(\pi\alpha(s)+\pi(\alpha(t))}{\sin\pi\alpha(s)} = \frac{e^{-i\pi(\alpha(s)+\alpha(t))} - e^{+i\pi(\alpha(s)+\alpha(t))}}{e^{-i\pi\alpha(s)} - e^{+i\pi\alpha(s)}}$$

$$\xrightarrow[s\to\infty]{} \frac{e^{-i\pi(\alpha(s)+\alpha(t))}}{e^{-i\pi\alpha(s)}} = e^{-i\pi\alpha(t)}$$

Therefore, the large s limit of $V(s,t,u)$ is: (off the real s-axis)

$$\boxed{V(s,t,u) \longrightarrow \frac{-\pi g}{\Gamma(1+\alpha(t))\sin\pi\alpha(t)} \underbrace{\left(1 + e^{-i\pi\alpha(t)}\right)}_{\text{only even signature contributes}} \alpha(s)^{\alpha(t)}}$$

$\left(\frac{s}{\alpha'^{-1}}\right)^{\alpha(t)}$

Gives the Regge scale factor $s_0 = 1/\alpha'$.