

Conservation Laws from Functional Methods

Context: Complex scalar field theory

$$\mathcal{L} = (\partial_\mu \phi)^* \partial^\mu \phi - m^2 \phi^* \phi$$

- invariant under $\phi \rightarrow e^{i\alpha} \phi$.

Allow $\alpha \rightarrow \alpha(x)$ to be a space-time dependent parameter. Implement such an infinitesimal transformation as a change of variables in the functional integral.

$$\phi(x) \rightarrow \phi(x) + i\alpha(x)\phi(x)$$

$$\int \mathcal{D}\phi \phi(x_1) \phi^*(x_2) e^{i \int d^4x \mathcal{L}[\phi]} = \int \mathcal{D}\phi' \phi'(x_1) \phi'^*(x_2) e^{i \int d^4x \mathcal{L}[\phi']}$$

Expand to first order in $\alpha(x)$ - leading terms cancel. $\partial \phi' = \partial \phi$ (Jac. det = 1)

$$0 = \int \mathcal{D}\phi \left\{ (i\alpha(x_1)\phi(x_1))\phi^*(x_2) - i\phi(x_1)(\alpha(x_2)\phi^*(x_2)) + \phi(x_1)\phi^*(x_2) i \int d^4x \partial_\mu \alpha (-i\phi^* \partial^\mu \phi) \right. \\ \left. + \phi(x_1)\phi^*(x_2) i \int d^4x \left[\partial_\mu \alpha (i(\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi)) \right] \right\} e^{i \int d^4x \mathcal{L}[\phi]}$$

↑
integrate by parts.

$$0 = \int \mathcal{D}\phi \int d^4x \alpha(x) \left\{ i\phi(x_1) \delta^4(x-x_2) \phi^*(x_2) - i\phi(x_1) \phi^*(x_2) \delta^4(x-x_2) + \phi(x_1)\phi^*(x_2) i \partial_\mu j^\mu(x) \right\} e^{i \int d^4x \mathcal{L}}$$

upon dividing
by Z

$$0 = i \langle 0 | T(\phi(x_1) \delta^4(x-x_2) \phi^*(x_2)) | 0 \rangle - i \langle 0 | T(\phi(x_1) \phi^*(x_2) \delta^4(x-x_2)) | 0 \rangle \\ + \langle 0 | T(\phi(x_1) \phi^*(x_2) i \partial_\mu j^\mu(x)) | 0 \rangle$$

or

$$\langle 0 | T(\phi(x_1) \phi^*(x_2) \partial_\mu j^\mu(x)) | 0 \rangle = \langle 0 | T(\phi(x_1) \delta^{(4)}(x-x_2) \phi^*(x_2)) | 0 \rangle$$

$$- \langle 0 | T(\phi(x_1) \phi^*(x_2) \delta^{(4)}(x-x_2)) | 0 \rangle$$

contact terms.

Take Fourier transform:

$$\text{Recall, } \tilde{G}^{[n]}(p_1, \dots, p_n) = \int d^4x_1 \dots d^4x_n e^{-ip_1 \cdot x_1 - \dots - ip_n \cdot x_n} G^{[n]}(x_1, \dots, x_n)$$

$$\text{Apply } \int d^4x_1 d^4x_2 d^4x e^{-ip_1 \cdot x_1 - ip_2 \cdot x_2 - iq \cdot x}$$

$$\begin{aligned} q_\mu \tilde{G}^{[3]}(p_1, p_2, q)^\mu &= \int d^4x_1 d^4x_2 e^{-ip_2 \cdot x_2 - i(p_1+q) \cdot x_1} \langle 0 | T(\phi(x_2) \phi^*(x_2)) | 0 \rangle \\ &\quad - \int d^4x_1 d^4x_2 e^{-ip_1 \cdot x_1 - i(p_2+q) \cdot x_2} \langle 0 | T(\phi(x_2) \phi^*(x_2)) | 0 \rangle \end{aligned}$$

$$q_\mu \tilde{G}^{[3]}(p_1, p_2, q)^\mu = \tilde{G}^{[2]}(p_1+q, p_2) - \tilde{G}^{[2]}(p_1, p_2+q)$$

This is the Ward identity for all Green's functions.