Summary of Generating Functionals

for Green's functions

generated by $\frac{\delta}{\delta(ij)}$

$$Z[j] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n \, G^{[n]}(x_1, \dots, x_n)\, ij(x_1) \dots ij(x_n)$$

for connected Green's functions

$$iW[j] = \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n \, G_c^{[n]}(x_1, \dots x_n)\, ij(x_1) \dots ij(x_n)$$

for one-particle-irreducible (proper) Green's functions

$$\Gamma[\phi_c] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n \, \Gamma^{[n]}(x_1, \dots x_n)\, \phi_c(x_1) \dots \phi_c(x_n)$$

Renormalization of Green's functions

All of the Green's functions appearing in the formulae above are (renormalization) scale-independent bare Green's functions. The definitions of the renormalized Green's functions are motivated by the renormalization of the field operators: $\phi_0 \equiv Z_\phi^{1/2} \phi_r$

$$Z[j] = \int \mathcal{D}\phi_0 \, e^{i\int d^4x \left(\mathcal{L}[\phi_0] + j\phi_0\right)}$$

$$= \int \mathcal{D}\phi_r \, e^{i\int d^4x \left(\mathcal{L}[\phi_r] + jZ_\phi^{1/2}\phi_r\right)}$$

So, when taking n derivatives of $Z[j]$ (and setting $j=0$), n factors of $Z_\phi^{1/2}$ come down.

$$Z[j] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n \, \underbrace{G^{[n]}(x_1, \dots, x_n)_{\text{Bare}}}_{\langle 0|\, T(\phi_0(x_1) \dots \phi_0(x_n))\,|0\rangle}\, ij(x_1) \dots ij(x_n)$$

$$= Z_\phi^{n/2} \langle 0|\, T(\phi_r(x_1) \dots \phi_r(x_n))\,|0\rangle$$

Define $\equiv G_R^{[n]}$

$$\boxed{G^{[n]}(x_1, \dots, x_n) = Z_\phi^{n/2}\, G^{[n]}(x_1, \dots, x_n)_R}$$

$$Z_r[j] = \sum_{n=0}^{\infty} \frac{1}{n!} Z_\phi^{n/2} \int d^4x_1 \dots d^4x_n \, G^{[n]}(x_1, \dots x_n)_R \; ij(x_1) \dots ij(x_n)$$

Renormalized connected Green's functions are defined in a similar way:

$$iW[j] = \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \, \underbrace{G_c^{[n]}(x_1, \ldots x_n)_{\text{Bare}}}_{\langle 0|T(\phi_0(x_1)\ldots\phi_0(x_n))|0\rangle_{\text{conn.}}} \, ij(x_1)\ldots ij(x_n)$$

$$= Z_\phi^{n/2} \langle 0|T(\phi_r(x_1)\ldots\phi_r(x_n))|0\rangle_{\text{conn.}} \longleftarrow \text{Define } G^{[n]}_{\text{conn}, R}$$

$$\boxed{G_c^{[n]}(x_1, \ldots x_n) = Z_\phi^{n/2} \, G_c^{[n]}(x_1, \ldots, x_n)_R}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} Z_\phi^{n/2} \int d^4x_1 \ldots d^4x_n \, G_c^{[n]}(x_1, \ldots, x_n)_R \, ij(x_1)\ldots ij(x_n)$$

For 1P-I Green's functions, use $\phi_c^{\text{Bare}} = Z_\phi^{1/2} \phi_c^R$. (not quite right)

$$\Gamma[\phi_c] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \, \Gamma^{[n]}_{\text{Bare}}(x_1, \ldots, x_n) \, \phi_c^{\text{Bare}}(x_1) \ldots \phi_c^{\text{Bare}}(x_n)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \, \underbrace{\Gamma^{[n]}_{\text{Bare}}(x_1, \ldots, x_n) \, Z_\phi^{n/2}}_{\equiv \Gamma_R^{[n]}(x_1, \ldots, x_n)} \, \phi_c^R(x_1) \ldots \phi_c^R(x_n)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \, \Gamma_R^{[n]}(x_1, \ldots, x_n) \, \phi_c^R(x_1) \ldots \phi_c^R(x_n)$$

note: renormalization scale dependence must cancel between Green's functions and the running fields, ϕ_c^R.

In other words:

$$\boxed{\Gamma_B^{[n]}(x_1, \ldots, x_n) = Z_\phi^{-n/2} \, \Gamma_R^{[n]}(x_1, \ldots, x_n)}$$

must be scale independent $\longrightarrow$ gives Callan-Symanzik equation.