Effective Potential for SHO.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \underbrace{+ J\phi}$$

$$L = \int d^3x \left[\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J\phi\right]$$

$$L = \frac{1}{2}\dot\phi^2 - \frac{1}{2}m^2\phi^2 + J\phi$$

[-1] [1].

$$\boxed{L = \frac{1}{2}\dot x^2 - \frac{1}{2}\omega^2 x^2 + Jx} \qquad e^{i\int d^4x\,\mathcal{L}} \qquad e^{i\int dt\,L}$$

$$p = \frac{\partial L}{\partial \dot x} = \dot x \qquad [x] = -\frac{1}{2},\ [p] = \frac{1}{2},\ [\omega] = 1,\ [J] = \frac{3}{2}$$

$\hat H = \frac{1}{2}\hat p^2 + \frac{1}{2}\omega^2\hat x^2 - J\hat x$ ← Solutions to this $\hat H|\psi_n\rangle = E_n|\psi_n\rangle$

$$= \frac{1}{2}\hat p^2 + \frac{1}{2}\omega^2\Big(\underbrace{\hat x - \frac{J}{\omega^2}}_{\hat X}\Big)^2 \underbrace{- \frac{1}{2}\omega^2\frac{J^2}{\omega^4}}_{-\frac{1}{2}\frac{J^2}{\omega^2}}$$

$$\hat H = \frac{1}{2}\hat p^2 + \frac{1}{2}\omega^2\hat X_J^2 - \frac{1}{2}\frac{J^2}{\omega^2}$$

$$\hat X_J = \frac{1}{\sqrt{2\omega}}\left(\hat a_J + \hat a_J^\dagger\right) \qquad \hat p = -i\sqrt{\frac{\omega}{2}}\left(\hat a_J - \hat a_J^\dagger\right) \qquad [\hat a] = 0.$$

unitless.

$$\Rightarrow \hat H = \omega\left(\hat a_J^\dagger a_J + \frac{1}{2}\right) - \frac{1}{2}\frac{J^2}{\omega^2}$$

States built out of $\hat a_J^\dagger$: $|1\rangle_J = \hat a_J^\dagger|0\rangle_J$

However, the true coordinate is not X; but x. related by $\hat x = \hat X + \frac{J}{\omega^2}$

So, $\underbrace{{}_J\langle 0|\hat x|0\rangle_J} = {}_J\langle 0|X + \frac{J}{\omega^2}|0\rangle_J = +\frac{J}{\omega^2}$

$$\left(\hat{X} - j/\omega^2\right)^2$$

$${}_J\langle 0|\hat{H}_{old}|0\rangle_J = {}_J\langle 0|\tfrac{1}{2}\hat{p}^2 + \tfrac{1}{2}\omega^2\hat{x}^2|0\rangle_J$$

$$= {}_J\langle 0|\tfrac{1}{2}(-i)^2\tfrac{\omega}{2}\underbrace{(\hat{a}_j - \hat{a}_j^\dagger)^2}_{\text{only one term here}} + \tfrac{1}{2}\omega^2\left[\tfrac{1}{\sqrt{2\omega}}\underbrace{(\hat{a}_j + \hat{a}_j^\dagger)}_{\text{only one term here}} + \tfrac{j}{\omega^2}\right]^2|0\rangle_J$$

$$= \frac{-\omega}{4}\,{}_J\langle 0| - \hat{a}_j\hat{a}_j^\dagger|0\rangle$$

$$+ \frac{1}{2}\omega^2\,{}_J\langle 0|\left(\frac{1}{2\omega}\underbrace{(\hat{a}_j + \hat{a}_j^\dagger)^2} + \frac{j^2}{\omega^4} + \frac{2}{\sqrt{2\omega}}\frac{j}{\omega^2}\underbrace{(\hat{a}_j + \hat{a}_j^\dagger)}_{\text{doesn't contribute}}\right)|0\rangle$$

$$= \frac{+\omega}{4} + \frac{\omega}{4}\,{}_J\langle 0|\hat{a}_j\cdot\hat{a}_j^\dagger|0\rangle_J + \frac{1}{2}\omega^2\frac{j^2}{\omega^4}\,{}_J\langle 0|0\rangle_J$$

$$= \frac{\omega}{2} + \frac{1}{2}\frac{j^2}{\omega^2}$$

Now eliminate j using $\dfrac{+j}{\omega^2} = {}_J\langle 0|\hat{x}|0\rangle_J$

$$+j = +\omega^2\,{}_J\langle 0|\hat{x}|0\rangle_J$$

$$= \frac{\omega}{2} + \frac{1}{2}\frac{1}{\omega^2}\omega^4\,{}_J\langle 0|\hat{x}|0\rangle_J^2$$

$$= \frac{\omega}{2} + \frac{1}{2}\omega^2\langle 0|\hat{x}|0\rangle_J^2$$

$$V_{eff}(\phi_{cl}) = \underbrace{\frac{1}{2}\omega^2 x_{cl}^2} + \frac{\omega}{2}$$

zero pt energy.

Equivalence of 2 def^{n}.

temperature

$$V(T,\bar{\phi}) = \lim_{V\to\infty} \frac{1}{V} \operatorname{Min}_{\rho} \operatorname{Tr}[\hat{\rho}\hat{H} + T\hat{\rho}\ln\hat{\rho}]$$

subject to $\operatorname{Tr}\hat{\rho} = 1$

$\operatorname{Tr}\hat{\rho}\hat{\phi} = \bar{\phi}$

Minimize with help of Lagrange multiplier.

Define:

mult #1   multp #2

$$L = \operatorname{Tr}[\hat{\rho}H + T\rho\ln\rho] + J\left(\operatorname{Tr}(\rho\phi) - \bar{\phi}\right) + K\left(\operatorname{Tr}(\rho) - 1\right)$$

Calculate δL. Use $\delta(\rho\ln\rho) = \delta\rho\ln\rho + \delta\rho$

$+ \delta J \operatorname{Tr}\rho\hat{\phi} - \bar{\phi}$

$+ \delta k (\operatorname{Tr}\hat{\rho} - 1) = 0$

(1) $\delta L = \operatorname{Tr}[\delta\hat{\rho}\hat{H} + T\delta\hat{\rho}\ln\hat{\rho} + T\delta\hat{\rho}] + J(\operatorname{Tr}\delta\hat{\rho}\hat{\phi}) + k\operatorname{Tr}\delta\hat{\rho}$

$$= \operatorname{Tr}\left[\underset{\neq 0}{\delta\rho}\left(\hat{H} + T\ln\hat{\rho} + T + J\hat{\phi} + k\right)\right] = 0$$

$\Rightarrow T\operatorname{Tr}\ln\hat{\rho} = \operatorname{Tr}[-\hat{H} - \hat{T} - J\hat{\phi} - k]$ and $\operatorname{Tr}\rho\hat{\phi} - \bar{\phi} = 0$

$\operatorname{Tr}\rho = 1$

$$\operatorname{Tr}\ln\hat{\rho} = \operatorname{Tr}\left[-\left(1 + \frac{k}{T}\right)\right] + \operatorname{Tr}\left[-\frac{1}{T}(\hat{H} + J\hat{\phi})\right]$$

A solution for $\hat{\rho}$ is:

$$\hat{\rho} = \underbrace{\exp\left[-\left(1 + \frac{k}{T}\right)\right]} e^{-\frac{1}{T}(\hat{H} + J\hat{\phi})}$$

$\operatorname{Tr}\hat{\rho} = 1 \Rightarrow \operatorname{Tr}\left[e^{-\frac{1}{T}(\hat{H} + J\hat{\phi})}\right]$

(non-trivial proof)

$\operatorname{Tr}\rho\hat{\phi} = \bar{\phi} \Rightarrow \operatorname{Tr}\hat{\rho}\hat{\phi} = -T\frac{\partial}{\partial J}\operatorname{Tr}\hat{\rho} = \bar{\phi}$

$$\cancel{\exp\left[A + xB\right] = \sum_{n=0}^{\infty} \frac{1}{n!} (A+xB)^n}$$

$$\cancel{\frac{d}{dx}\exp\left[\quad\right] = \sum_{n=0}^{\infty} \frac{1}{n!}}$$

Take $T \to 0$ limit:

$$V(T,\bar{\phi}) = \lim_{V\to\infty} \min_{\rho} \mathrm{Tr}\left[\rho \hat{H} - T\hat{\rho}\ln\hat{\rho}\right] \qquad \text{solution: } \hat{\rho} = \frac{e^{-\frac{1}{T}(\hat{H}+J\hat{\phi})}}{\mathrm{Tr}[\quad]}$$

How does solution behave in $T \to 0$ limit?

$$\hat{\rho} = \frac{1}{Z} e^{-\frac{1}{T}(\hat{H}+J\hat{\phi})}$$

$$= \frac{1}{Z}\sum_{n=0}^{\infty} e^{-\lambda_n/T} |\psi_n\rangle\langle\psi_n| \qquad Z = \sum_n e^{-\lambda_n/T}$$

If state with lowest E-value exists, can take $T\to 0$ limit easily

$$\to \frac{e^{-\lambda_0/T}|\psi_0\rangle\langle\psi_0| + e^{-\lambda_1/T}|\psi_1\rangle\langle\psi_1| + \ldots}{e^{-\lambda_0/T} + e^{-\lambda_1/T} + \ldots}$$

$$= |\psi_0\rangle\langle\psi_0|$$

Since solution is of this form, can restrict variational problem to this.

$$V(0,\bar{\phi}) = \lim_{V\to\infty} \frac{1}{V} \min_{\rho} \mathrm{Tr}(\rho\hat{H}) \quad \overset{\text{equiv.}}{\equiv} \quad V(0,\bar{\phi}) = \lim_{V\to\infty}\frac{1}{V}\min_{|\psi\rangle}\langle\psi|\hat{H}|\psi\rangle$$

$\mathrm{Tr}\,\rho = 1$ | subject to:
$\mathrm{Tr}\,\rho\hat{\phi} = \bar{\phi}$ | $\langle\psi|\psi\rangle = 1$
$\rho \propto |\psi\rangle\langle\psi|$ for some ψ. | $\langle\psi|\hat{\phi}|\psi\rangle = \bar{\phi}$