Effective Potential from Effective action

Evaluate effective action at Homogeneous field configuration

$$\phi(x) = \phi$$

Then

$$\Gamma[\phi_c] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \, \Gamma^{[n]}(x_1, \ldots x_n; \mu) \, \phi_c^{\,n}$$

to momentum space:

$$\Gamma^{[n]}(x_1, \ldots, x_n) = \int \frac{d^4k_1}{(2\pi)^d} \cdots \frac{d^4k_n}{(2\pi)^d} (2\pi)\, \delta^{(4)}(k_1 + \ldots + k_n)\, e^{-i(k_1\cdot x_1 + \ldots k_n \cdot x_n)}\, \tilde{\Gamma}^{[n]}(k_1, \ldots k_n)$$

Integrate over all the x's

$$\int d^4x_1 \, e^{-i(k_1 \cdot x_1)} = (2\pi)^4 \delta^{(4)}(k_1)$$

$$\vdots$$

$$\int d^4x_n \, e^{-i(k_n \cdot x_n)} = (2\pi)^4 \delta^{(4)}(k_n)$$

Integrate over all the k's using these new delta functions.

fixing $k_1, \ldots k_n \to 0$.

$$\Gamma[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{(2\pi)^4 \delta^{(4)}(0)}_{=\frac{1}{V_4}} \tilde{\Gamma}^{[n]}(k_1 = 0, \ldots, k_n = 0) \, \phi_c^{\,n}$$

$$= \frac{1}{V_4} \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\Gamma}^{[n]}(0, \ldots, 0)\, \phi_c^{\,n} \overset{\text{define:}}{=} \frac{1}{V_4}\left(-V_{eff}(\phi)\right)$$

$$V_{eff}(\phi) = -\sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\Gamma}^{[n]}(0, \ldots, 0)\, \phi^n$$