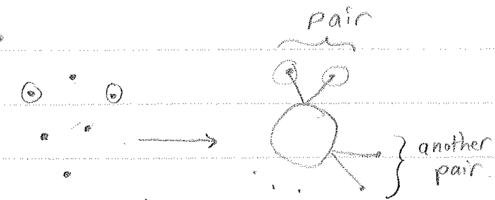
Symmetry Factors

n = number of internal vertices
$2n$ = number of ext. fields.

- Count the number of ways to pair up external fields:

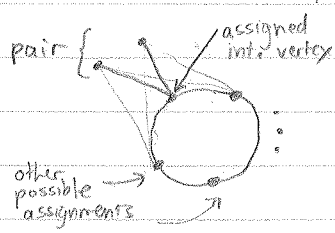


(2n external points)

$$= (2n-1)(2n-3)\dots 1$$

$$= \frac{(2n-1)(2n-2)(2n-3)\dots 1}{(2n-2)(2n-4)\dots} = \frac{(2n-1)!}{2(n-1)\,2(n-2)\dots} = \boxed{\frac{(2n-1)!}{2^{n-1}(n-1)!}}$$

- Count number of ways to assign each pair of fields to an internal vertex



$$(\text{possible number of assignments}) = \boxed{n!}$$

So we have $\underbrace{(\phi\phi)}\ \underbrace{(\phi\phi)}\dots\ \underbrace{\phi\phi\phi\phi}\ \ \underbrace{\phi\phi\phi\phi}\dots$

paired fields

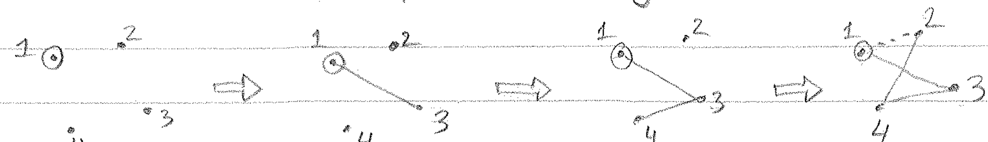
assigned pair of fields to vertices.

- For each pair of field to vertex assignments
  - First field can contract into one of 4 int. vertex fields eg $(\phi\phi)\ \ \phi\phi\phi\phi$
  - Second field can contract to any one of remaining 3 int. vertex fields. eg $(\phi\phi)\ \ \phi\phi\phi\phi$

$$\underbrace{(4\times3)\times(4\times3)\times\dots(4\times3)}_{\text{number of internal vertices}} = \boxed{(4\times3)^n}$$

Now we need to contract the internal fields to build the ring.

- Count number of distinct ways to build ring:



1st vert. has 3 ch. — 2nd vertex has 2 ch. — 3rd vertex has 1 ch. — (1-3-4-2).

Generalize to n int. vertices: $\frac{(n-1)(n-2)\dots 1}{2} = \boxed{\frac{(n-1)!}{2}}$ ring built clockwise is same as counterclockwise

EXCEPT for $n=2$, is actually $(n-1)!$

contractions to ext. pairs

• Count the total internal contractions: $\underbrace{\phi\phi\phi\phi}_{1} \longrightarrow \underbrace{\phi\phi\phi\phi}_{3} \longrightarrow \underbrace{\phi\phi\phi\phi}_{4} \longrightarrow \underbrace{\phi\phi\phi\phi}_{2}$

From vertex 1, either of the two fields can contract into either of the two fields in vertex 3. $= 2\times 2$.

The remaining field contracts into either of the two fields in vertex 4 $= \times 2$

⋮

$$\underbrace{(2\times 2)\times 2\times 2\times \ldots \times 2}_{n \text{ int. vertices.}} = \boxed{2^n}$$

EXCEPT for $n=2$, just 2^{n-1} possible contractions

So, putting everything together, (the $n=2$ case, and $n>2$ cases merge b/c $\frac{1}{2}\times\cancel{2}$). (exceptions cancel)

$$\frac{1}{(2n)!}\underbrace{\left[\frac{(2n-1)!}{2^{n-1}(n-1)!}\times n!\times(4\times 3)^n\times\frac{(n-1)!}{2}\times 2^n\right]}_{\text{calculated symmetry factor}}\times\frac{1}{n!}\times\left(\frac{-i\lambda}{4!}\right)^n$$

Def^n of Eff. pot   Taylor expansion   Feynman Rule (incl. its combinatoric factor)

$$= \frac{1}{(2n)!}\,\frac{(2n)!}{2^n\,2n}$$

Then, using $V(\phi_c) = -\sum_n \frac{1}{(2n)!}\Gamma^{[2n]}(p_1=0,\ldots 0)\,\phi_c^{2n}$, we have:

$$V(\phi_c) = \frac{1}{2}m_R\phi_c^2 + \frac{\lambda_R}{4!}\phi_c^4 \underbrace{- \sum_n \frac{1}{(2n)!}(-i)\frac{(2n)!}{2^n\,2n}\int\frac{d^dp}{(2\pi)^d}\left[(-i\lambda_R\mu^{2\epsilon})\frac{i}{p^2-m_R^2+i\epsilon}\right]^n\phi_c^{2n}}_{V_1(\phi_c)}$$

$$V_1(\phi_c) = i\sum_{n=1}^{\infty}\int\frac{d^dp}{(2\pi)^d}\,\frac{1}{2n}\left(\frac{\lambda_R\mu^{2\epsilon}\phi^2/2}{p^2-m_R^2+i\epsilon}\right)^n$$

$\Gamma^{[2n]}(0,\ldots,0)$ has dimensions $(4-2n)+(-2+2n)\epsilon$. Pull out $(-2+2n)$ factors of μ^ϵ, and keep remainder $\mu^{2\epsilon}$ to expand in the end.

$$V_1(\phi_c) = i\sum_{n=1}^{\infty}\underbrace{\mu^{(-2+2n)\epsilon}}_{\text{Immediately take } \epsilon\to 0 \text{ lim.}}\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\,\frac{1}{2n}\left(\frac{\lambda_R\phi^2/2}{p^2-m_R^2+i\epsilon}\right)^n$$

$$V_1(\phi_c) = i\mu^{2\epsilon} \sum_{n=1}^{\infty} \int \frac{d^dp}{(2\pi)^d} \frac{1}{2n} \left(\frac{\lambda_R \phi_c^2/2}{p^2 - m_R^2 + i\epsilon} \right)^n + \delta_{ZP} + \frac{1}{2}\delta_{m^2}\phi^2 + \frac{1}{4!}\delta_\lambda \phi^4$$

$$= -\frac{i}{2}\mu^{2\epsilon} \int \frac{d^dp}{(2\pi)^d} \ln\left(1 - \frac{\lambda\phi_c^2/2}{p^2 - m^2 + i\epsilon}\right)$$

$$= -\frac{i}{2}\mu^{2\epsilon} \int \frac{d^dp}{(2\pi)^d} \ln\left(\frac{p^2 - m^2 - \lambda\phi_c^2/2 + i\epsilon}{p^2 - m^2 + i\epsilon}\right)$$

Split log (separate ϕ-dependent part from the ϕ-indep. part), introduce arbitrary mass $\tilde{M}$ — will cancel later.

$$= -\frac{i}{2}\left[\mu^{2\epsilon} \int \frac{d^dp}{(2\pi)^d} \ln\left(\frac{p^2 - m^2 - \lambda\phi_c^2/2 + i\epsilon}{\tilde{M}^2}\right) - \mu^{2\epsilon} \int \frac{d^dp}{(2\pi)^d} \ln\left(\frac{p^2 - m^2 + i\epsilon}{\tilde{M}^2}\right)\right]$$

Abbreviate $m^2 + \lambda\phi_c^2/2 = m_s^2(\phi)$. Integrate by introducing $\frac{d}{dm_s^2}$

$$= -\frac{i}{2} \int dm_s^2\, \mu^{2\epsilon} \int \frac{d^dp}{(2\pi)^d} \frac{d}{dm_s^2} \ln\left(\frac{p^2 - m_s^2(\phi) + i\epsilon}{\tilde{M}^2}\right) - (\text{field indep.})$$

Tadpole method yields this expression →

$$= +\frac{i}{2} \int dm_s^2\, \mu^{2\epsilon} \int \frac{d^dp}{(2\pi)^d} \frac{\cancel{\tilde{M}^2}}{p^2 - m_s^2(\phi) + i\epsilon} \frac{1}{\cancel{\tilde{M}^2}} - (\text{field indep.})$$

$$= +\frac{i}{2} \int dm_s^2\, \mu^{2\epsilon} \frac{(-1)\, i}{(4\pi)^{d/2}} \frac{\Gamma(1 - d/2)}{\Gamma(1)} \left(\frac{1}{m_s^2(\phi) - i\epsilon}\right)^{1 - d/2} - (\text{field indep.})$$

Integrate over m_s^2:

$$= +\frac{i}{\cancel{2}}\, \mu^{2\epsilon} \frac{(-1)\, i}{(4\pi)^{d/2}} \frac{\Gamma(1 - d/2)}{\Gamma(1)} \frac{1}{d/\cancel{2}} \left(\frac{1}{m_s^2(\phi) - i\epsilon}\right)^{-d/2}$$

$$= \frac{1}{16\pi^2} (4\pi\mu^2)^\epsilon \frac{\Gamma(-1 + \epsilon)}{4 - 2\epsilon} \frac{1}{(m_s^2(\phi) - i\epsilon)^{-2+\epsilon}}$$

$$= -\frac{1}{2} \frac{1}{2(4\pi)^2} \left[m_s^2(\phi)\right]^2 \left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi - \ln\left(\frac{m_s^2(\phi) - i\epsilon}{\mu^2}\right) + \frac{3}{2}\right)$$

$$- \left(\begin{matrix}\text{Field Indep} \\ \text{part}\end{matrix}\right) + \delta_{ZP} + \frac{1}{2}\delta_{m^2}\phi_c^2 + \frac{1}{4!}\delta_\lambda \phi^4.$$

$$V_1(\phi_c) = \frac{-1}{64\pi^2}\left(m_R^4 + \lambda_R m_R^2 \phi_c^2 + \lambda_R^2 \phi_c^4/4\right)\left(\alpha_\epsilon - \ln\left(\frac{m^2+\lambda_R\phi_c^2/2 - i\epsilon}{\mu^2}\right) + \frac{3}{2}\right)$$

$$+ \begin{pmatrix}\text{Field indep.}\\ \text{part}\end{pmatrix} + \delta_{ZP} + \frac{1}{2}\delta_{m^2}\phi_c^2 + \frac{1}{4!}\delta_{\phi^4}\phi^4$$

In $\overline{\text{MS}}$ scheme, corresponding counterterms cancel divergences in each term.

$$V_1^{\overline{\text{MS}}}(\phi_c) = \frac{+1}{64\pi^2}\left(m_R^4 + \lambda_R m_R^2 \phi_c^2 + \lambda_R^2 \phi_c^4/4\right)\left(\ln\left(\frac{m^2+\lambda_R\phi_c^2/2 - i\epsilon}{\mu^2}\right) - \frac{3}{2}\right)$$

$$+ \begin{pmatrix}\text{Field indep.}\\ \text{Part}\end{pmatrix}.$$

The field independent part merely contributes to the zero point energy.

Derived Formula:

$$\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\ln(p^2-m^2) = \frac{-i}{2(4\pi)^2}(m^2)^2\left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi - \ln\left(\frac{m^2 - i\epsilon}{\mu^2}\right) + \frac{3}{2}\right)$$