Effective Potential of the Yukawa Theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \bar{\psi}(i\not{\partial} - M)\psi - g\phi\bar{\psi}\psi$$

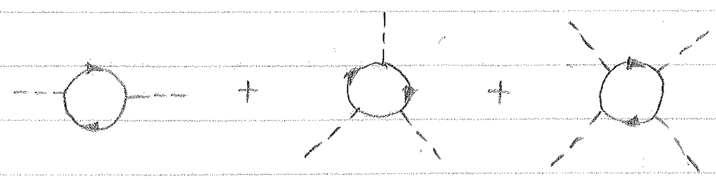
Renormalize (Dim. Reg. — see back)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m_R^2\phi^2 - \frac{\lambda_R}{4!}\mu^{2\epsilon}\phi^4 + \bar{\psi}(i\not{\partial} - M_R)\psi - g_R\mu^{\epsilon}\bar{\psi}\psi$$

$$+\frac{1}{2}\delta_\phi\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\delta_{m^2}\phi^2 - \frac{\delta_\lambda\mu^{2\epsilon}}{4!}\phi^4 + \underbrace{\delta_\psi\bar{\psi}(i\not{\partial})\psi - \delta_M\bar{\psi}\psi}_{\text{Not needed}} - \delta_g\mu^{\epsilon}\phi\bar{\psi}\psi$$

— we will find many more terms are needed.

Diagrams to sum:

$V^{1\text{-loop}}_{\text{Fermion}}(\phi_c) =$  $+ \ldots$

Symmetry factors: (Let n = number of internal vertices)

- Number of ways to assign each external field to an internal vertex: $\boxed{= n!}$
- Number of ways to build fermion ring:

This time, $\phi\bar{\psi}\psi\ \phi\bar{\psi}\psi\ \phi\bar{\psi}\psi \ldots \phi\bar{\psi}\psi$ is a different contraction from $\phi\bar{\psi}\psi\ \phi\bar{\psi}\psi\ \phi\bar{\psi}\psi \ldots \phi\bar{\psi}\psi$,

so no 1/2 factor to prevent double counting

$\Rightarrow \boxed{(n-1)!}$

So, putting everything together, the total numeric prefactor is

$$\underset{\substack{\uparrow\\ \text{Def}^{\text{n}}\\ \text{eff. pot}}}{\frac{1}{n!}}\underbrace{\left[n!\,(n-1)!\right]}_{\text{calculated symm. factor}}\ \underset{\substack{\uparrow\\ \text{Taylor}\\ \text{exp}}}{\frac{1}{n!}}\underbrace{(-ig\mu^{2\epsilon})^n}_{\text{Feyn. Rule.}} = \frac{1}{n!}(n-1)!\,(-ig\mu^{2\epsilon})^n$$

Then,

$$V^{1\text{-loop}}_{\text{Fermion}}(\phi_c) = \sum_{n=1}^{\infty}\underbrace{\frac{1}{n!}(-i)(n-1)!}_{\frac{1}{n}}\underset{\substack{\uparrow\\ \text{Fermion}\\ \text{Loop}}}{(-1)}\int\frac{d^dp}{(2\pi)^d}(-ig_R\mu^{\epsilon})^n\frac{i^n\,\overbrace{\mathrm{Tr}[(\not{p}+M_R)\ldots]}^{n\text{ factors}}}{(p^2-M_R^2+i\epsilon)^n}\phi_c^{\,n}$$

Use $\mathrm{Tr}\Big[\underbrace{(\not p+M)\ldots(\not p+M)}_{n\text{ copies}}\Big] = \frac{1}{2}\Big[(\sqrt{p^2}+M)^n + (-\sqrt{p^2}+M)^n\Big]\,\mathrm{Tr}[\mathbb{1}]$

Then $V^{1\text{-loop}}_{\text{fermion}}(\phi_c) = +i\sum_{n=1}^{\infty}\frac{1}{n}\int\frac{d^dp}{(2\pi)^d}\,\frac{(g_R\mu^{\epsilon})^n}{(p^2-M_R^2+i\epsilon)^n}\,\frac{1}{2}\Big[(\sqrt{p^2}+M_R)^n+(-\sqrt{p^2}+M_R)^n\Big]\,\mathrm{Tr}[\mathbb{1}]\,\phi_c^n$

Calculate both cases together as $(\pm\sqrt{p^2}+M_R)^n$. Then add them later

Dimension of $\Gamma^{[n]}$ is $(4-n)+(-2+n)\epsilon$

Pull out $(-2+n)$ factors of μ^{ϵ}.

$$V_1(\phi_c) = i\sum_{n=1}^{\infty}\frac{1}{n}\underbrace{\mu^{(-2+n)\epsilon}}_{\text{set }\epsilon\to 0}\,\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\,\frac{1}{2}\sum_{\pm}\left[\frac{g_R\phi_c(\pm\sqrt{p^2}+M_R)}{p^2-M_R^2+i\epsilon}\right]^n \mathrm{Tr}[\mathbb{1}]$$

Perform sum over n

$$V_1(\phi_c) = -i\,\mathrm{Tr}[\mathbb{1}]\,\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\,\frac{1}{2}\sum_{\pm}\ln\Bigg(1-\frac{g_R\phi_c(\pm\sqrt{p^2}+M_R)}{\underbrace{p^2-M_R^2+i\epsilon}_{\text{write as } -(\sqrt{p^2}+M_R^2)(-\sqrt{p^2}+M_R^2)+i\epsilon}}\Bigg)$$

$$= -i\,\mathrm{Tr}[\mathbb{1}]\,\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\,\frac{1}{2}\sum_{\pm}\ln\left(1+\frac{g_R\phi_c}{+(\mp\sqrt{p^2}+M_R)+i\epsilon}\right)$$

$$= -i\,\mathrm{Tr}[\mathbb{1}]\,\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\,\frac{1}{2}\sum_{\pm}\ln\left(\frac{\mp\sqrt{p^2}+M_R+g_R\phi_c+i\epsilon}{\mp\sqrt{p^2}+M_R+i\epsilon}\right)$$

Split into two logs.

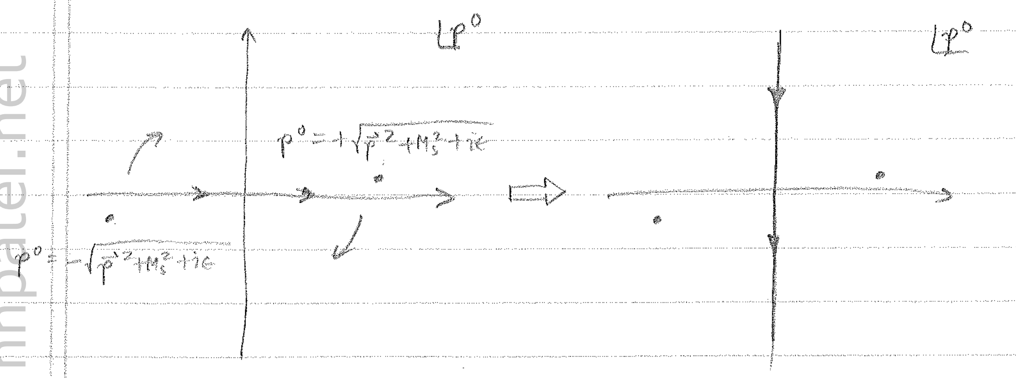
$$= -i\,\mathrm{Tr}[\mathbb{1}]\left[\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\,\frac{1}{2}\sum_{\pm}\ln\left(\frac{\mp\sqrt{p^2}+\overbrace{M_R+g_R\phi_c}^{M_S(\phi_c)}+i\epsilon}{\tilde{M}}\right)-\underbrace{\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\,\frac{1}{2}\sum_{\pm}\ln\left(\frac{\mp\sqrt{p^2}+M_R}{\tilde{M}}\right)}_{\text{Field indep.}}\right]$$

Abbreviate $M_R+g_R\phi_c \equiv M_S(\phi_c)$ "shifted mass". Integrate by introducing $\frac{d}{dM_S}$

$$= -i\,\mathrm{Tr}[\mathbb{1}]\int dM_S\,\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\,\frac{1}{2}\sum_{\pm}\frac{d}{dM_S}\ln\left(\frac{\mp\sqrt{p^2}+M_S(\phi_c)+i\epsilon}{\tilde{M}}\right)\ldots$$

$$V_1(\phi_c) = -i\,\mathrm{Tr}[\mathbb{1}] \int dM_s\, \mu^{2\epsilon} \int \frac{d^dp}{(2\pi)^d}\, \frac{1}{2} \sum_{\pm} \frac{1}{\mp\sqrt{\vec{p}^{\,2} + M_s(\phi_c)} + i\epsilon} - \ldots$$

(anti-)Wick Rotate. Poles structure same for both $\pm$ terms.



Limits: p^0: $+i\infty \to -i\infty$

Redefine $p^0 = -ip_E^0$

New Limits: p_E^0: $-\infty \to +\infty$

$$\int \frac{d^dp}{(2\pi)^d}\, \frac{1}{2} \sum_{\pm} \frac{1}{\mp\sqrt{p^2} + M_s + i\epsilon} = \int \frac{d^d(-ip_E)}{(2\pi)^d}\, \frac{1}{2} \sum_{\pm} \frac{1}{\mp\sqrt{-p_E^2} + M_s + i\epsilon}$$

$\longrightarrow$ To hyperspherical coordinates.

$$= -i \int \frac{d\Omega_d}{(2\pi)^d} \int d|p_E|\, \frac{1}{2} \sum_{\pm} \frac{|p_E|^{d-1}}{\mp i|p_E| + M_s} \qquad \text{open up sum.}$$

$$= -i \int \frac{d\Omega_d}{(2\pi)^d} \int d|p_E|\, \frac{1}{2} \left[\frac{|p_E|^{d-1}}{i|p_E| + M_s} + \frac{|p_E|^{d-1}}{-i|p_E| + M_s} \right]$$

$$= -i \int \frac{d\Omega_d}{(2\pi)^d} \int d|p_E|\, \frac{1}{\cancel{2}}\, \cancel{2}M_s \left[\frac{|p_E|^{d-1}}{|p_E|^2 + M_s^2} \right]$$

$$= -iM_s \int \frac{d\Omega_d}{(2\pi)^d} \int d|p_E|\, \frac{|p_E|^{d-1}}{|p_E|^2 + M_s^2}$$

Use Gradshteyn (1980) Eq. 3.241(4) on pg. 292 to evaluate $d|p_E|$ integral

$$= -iM_s\, \frac{1}{(2\pi)^d}\, \frac{2(\sqrt{\pi})^d}{\Gamma(d/2)}\, \frac{1}{2M_s^2} \left(M_s^2\right)^{d/2} \frac{\Gamma(\frac{d}{2})\Gamma(1-\frac{d}{2})}{\Gamma(1)}$$

$$= -i \left[\frac{1}{(4\pi)^{d/2}}\, \Gamma\!\left(1-\frac{d}{2}\right) \left(M_s^2\right)^{\frac{d-1}{2}} \right]$$

Plug into $V_1(\phi_c)$, and integrate over $dM_s(\phi)$.

$$V_1(\phi_c) = \underbrace{-i}_{\text{cancel}} \mathrm{Tr}[\mathbb{1}] \int dM_s \, \mu^{2\epsilon} \left[\underbrace{-i}_{} \frac{1}{(4\pi)^{d/2}} \Gamma\left(1-\frac{d}{2}\right) (M_s^2)^{(d-1)/2} \right] - \ldots$$

$$= -\mathrm{Tr}[\mathbb{1}] \frac{\mu^{2\epsilon}}{(4\pi)^{d/2}} \frac{1}{d} \Gamma\left(1-\frac{d}{2}\right) \left[M_s^2(\phi)\right]^{d/2}$$

Expand around $\epsilon \approx 0$:

$$= +\mathrm{Tr}[\mathbb{1}] \frac{[M_s^2(\phi_c)]^2}{64\pi^2} \left(\frac{1}{\epsilon} - \gamma_E + \ln(4\pi) - \ln\left(\frac{M_s^2(\phi_c)}{\mu^2}\right) + \frac{3}{2} \right)$$

$$+ \begin{pmatrix}\text{Field}\\ \text{indep.}\end{pmatrix} + \delta_{ZP} + \delta_\phi \phi + \frac{\delta_{m^2}}{2}\phi^2 + \frac{\delta_{\phi^3}}{3!}\phi^3 + \frac{\delta_\lambda}{4!}\phi^4$$

$$= \mathrm{Tr}[\mathbb{1}] \frac{1}{64\pi^2} \left(M_R^4 + 4 g_R M_R^3 \phi_c + 6 g_R^2 M_R^2 \phi_c^2 + 4 g_R^3 M_R \phi_c^3 + g_R^4 \phi_c^4 \right)$$

$$\times \left(a_\epsilon - \ln\left(\frac{M_s^2(\phi_c)}{\mu^2}\right) + \frac{3}{2} \right) + \begin{pmatrix}\text{Field}\\ \text{indep.}\end{pmatrix} + \text{counterterms.}$$

$\mathrm{Tr}[\mathbb{1}] = 4$, $\quad M_s(\phi_c) = M + g\phi_c$

Discrete chiral symmetry

In the limit $M \to 0$, Yukawa theory has a "discrete chiral symetry":

$$\phi \to -\phi \qquad \psi \to e^{\frac{i\pi}{2}\gamma_5} \psi \qquad \bar{\psi} \to \bar{\psi} e^{+\frac{i\pi}{2}\gamma_5}$$

Yukawa coupling term invariant:

$$-g\phi\bar{\psi}\psi \longrightarrow -g(-\phi)\bar{\psi} \underbrace{\left(e^{\frac{i\pi}{2}\gamma_5}\right)\left(e^{\frac{i\pi}{2}\gamma_5}\right)}_{-1} \psi = -g\phi\bar{\psi}\psi.$$

$\Rightarrow$ No δ_ϕ and $\delta_{\phi\phi\phi}$ counterterms necessary.