## Path Integral Approach to Effective Potentials ($\phi^4$ theory)

Start with $Z[J] = \int \mathcal{D}\phi \; e^{i\int d^4x[\mathcal{L}(\partial\phi,\phi)+j\phi]}$,

$$\text{where } S[\phi,j] = \int d^4x \left(\frac{1}{2}(1+\delta_\phi)\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}(m^2+\delta_{m^2})\phi^2 - \frac{1}{4!}(\lambda+\delta_\lambda)\phi^4 + j\phi\right)$$

is the renormalized action. We are interested in developing an effective theory for $\phi_c$, the classical (background) field, by integrating over the quantum fluctuations. First, split $\phi(x) = \phi_c + \phi'(x)$, where $\phi_c$ is independent of spacetime, and $\phi'(x)$ are the quantum fluctuations, with $\langle\phi'\rangle = 0$.

$$\text{So now, } S[\phi,j] = \int d^4x\Big[\frac{1}{2}(1+\delta_\phi)\partial_\mu\phi'\partial^\mu\phi' - \frac{1}{2}(m^2+\delta_{m^2})(\phi_c+\phi')^2 - \frac{1}{4!}(\lambda+\delta_\lambda)(\phi_c+\phi')^4 + j(\phi_c+\phi')\Big]$$

The functional integration measure then changes, $\mathcal{D}\phi \to \mathcal{D}\phi'$. Organize the action in powers of $\phi'(x)$.

$$\begin{aligned}
S[\phi_c,\phi',j] = \int d^4x\Big[ & -\frac{1}{2}(m^2+\delta_{m^2})\phi_c^2 - \frac{1}{4!}(\lambda+\delta_\lambda)\phi_c^4 + j\phi_c && \longleftarrow \mathcal{O}(\phi')^0 \\
& -(m^2+\delta_{m^2})\phi_c\phi' - \frac{1}{6}(\lambda+\delta_\lambda)\phi_c^3\phi' + j\phi' && \longleftarrow \mathcal{O}(\phi')^1 \\
& +\frac{1}{2}(1+\delta_\phi)\partial_\mu\phi'\partial^\mu\phi' - \frac{1}{2}(m^2+\delta_{m^2})\phi'^2 - \frac{1}{4}\phi_c^2\phi'^2 && \longleftarrow \mathcal{O}(\phi')^2 \\
& -\frac{1}{4!}(\lambda+\delta_\lambda)\phi'^4 - \frac{1}{6}(\lambda+\delta_\lambda)\phi_c\phi'^3\Big] && \longleftarrow \text{rest.}
\end{aligned}$$

Since $\phi_c$ is independent of space-time, the $\int d^4x$ integral can be performed in the $\mathcal{O}(\phi')^0$ terms immediately. Push the counterterms in $\mathcal{O}(\phi')^1$ & $\mathcal{O}(\phi')^2$ to the "rest".

$$\begin{aligned}
S[\phi_c,\phi',j] = {} & V\left(-\frac{1}{2}m^2\phi_c^2 - \frac{1}{4!}\phi_c^4 - \frac{1}{2}\delta_{m^2}\phi_c^2 - \frac{1}{4!}\delta_\lambda\phi^4 + j\phi_c\right) && \leftarrow \mathcal{O}(\phi')^0 \\
& + \left(-m^2\phi_c - \frac{\lambda}{6}\phi_c^3 + j\right)\int d^4x\;\phi'(x) && \longleftarrow \mathcal{O}(\phi')^1 \\
& + \int d^4x\left[\frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{1}{2}\left(m^2+\frac{\lambda}{2}\phi_c^2\right)\phi'^2\right] && \leftarrow \mathcal{O}(\phi')^2 \\
& + \int d^4x\Big[-\frac{1}{4!}\lambda\phi'^4 - \frac{1}{6}\lambda\phi_c\phi'^3 - \delta_{m^2}\phi_c\phi' - \frac{1}{6}\delta_\lambda\phi_c^3\phi' && \Big\} \text{ Everything} \\
& \qquad - \frac{1}{4!}\delta_\lambda\phi'^4 - \frac{1}{6}\delta_\lambda\phi_c\phi'^3\Big] && \Big\} \text{ else.}
\end{aligned}$$

Since the $\mathcal{O}(\phi')^0$ is independent of $\phi'$, abbreviate it as $S[\phi_c]$.

In the $\mathcal{O}(\phi')^1$ term, abbreviate $\tilde{j} \equiv -m^2\phi_c - \frac{\lambda}{6}\phi_c^3$ as the effective source for $\phi'(x)$. $\;\;\overset{\shortparallel}{\;}\; S_2[\phi', \phi_c]$

In the $\mathcal{O}(\phi')^2$ term, abbreviate $m_S^2(\phi_c) \equiv m^2 + \frac{\lambda}{2}\phi^2$ as the shifted mass.

Then, $Z[j] = e^{iS[\phi_c]}\int \mathcal{D}\phi' \, e^{i(\tilde{j}\int d^4x\,\phi' + S_2[\phi_c,\phi'] + S_{int}[\phi_c,\phi'])}$,

where $S_2[\phi_c, \phi'] = \int d^4x \left[\frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{1}{2}\underbrace{(m^2 + \frac{\lambda}{2}\phi_c^2)}_{m_S^2(\phi_c)}\phi'^2\right]$

and $S_{int}[\phi_c, \phi'] = \int d^4x \left[-\frac{1}{4!}\lambda\phi'^4 - \frac{1}{6}\lambda\phi_c\phi'^3 + (\text{counterterms})\right]$.

Since $\frac{\partial Z}{\partial \tilde{j}} = \langle\phi'\rangle \equiv 0$, the linear term in $Z[j]$ must vanish. So, we have

$$Z[j] = e^{iS[\phi_c]}\underbrace{\int \mathcal{D}\phi' \, e^{i\int d^4x\left[\frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{1}{2}m_S^2(\phi_c)\phi'^2\right] + iS_{int}[\phi_c,\phi']}}_{\text{The integral over this is straight-forward} \doteq [\mathrm{Det}(\mathcal{O})]^{-1/2}}.$$

$$= e^{iS[\phi_c]}\left[\mathrm{Det}(\partial^2 + m_S^2(\phi_c))\right]^{-1/2}\int \mathcal{D}\phi \, e^{iS_{int}[\phi_c,\phi']}$$

Recall, the relationship between the generating functional (for Green's functions), and the quantum action (generating functional for 1PI, or proper diagrams), which contains the effective potential.

$$\underset{\substack{\uparrow \\ \text{Quantum Action}}}{\Gamma(\phi_c)} = \underbrace{-\int d^4x\, j\phi_c - i\ln Z[j]}_{(\text{Legendre transform})} \equiv \underbrace{\int d^4x\left[-V_{eff}(\phi_c) + \ldots\right]}_{(\text{momentum expansion})}.$$

Substituting in $Z[j]$ into above expression, we get

$$\underbrace{-\int d^4x\, j\phi_c}_{\nabla j\phi_c} - i\ln\left[e^{iS[\phi_c]}\left[\mathrm{Det}(\partial^2 + m_S^2(\phi))\right]^{-1/2}\int \mathcal{D}\phi' \, e^{iS_{int}[\phi_c,\phi']}\right] = -\nabla V_{eff} \underbrace{+\ldots}_{\substack{\text{dropped since } \phi_c \\ \text{in indep. of spacetime} \\ \longrightarrow \text{only zero-momentum}}}$$

— cancel minus sign.

$$\mathcal{V} j \phi_c \quad + \underbrace{i\,(\,i}_{\ominus} S[\phi_c]) + i \ln\left[\mathrm{Det}\left(\partial^2 + m_S^2(\phi)\right)\right]^{-1/2}$$

$$i \ln\left[\int \mathcal{D}\phi' \, e^{iS_{int}[\phi_c, \phi']}\right] = \mathcal{V}\, V_{eff}(\phi_c)$$

$$\cancel{\mathcal{V} j \phi_c} - \mathcal{V}\left(-\frac{1}{2} m^2 \phi_c^2 - \frac{1}{4!}\phi_c^4 - \frac{1}{2}\delta_{m^2}\phi_c^2 - \frac{1}{4!}\delta_\lambda \phi_c^4 + \cancel{j\phi_c}\right)$$

$$+ \frac{i}{2} \ln \mathrm{Det}\left(\partial^2 + m_S^2(\phi)\right) + i \ln\left[\int \mathcal{D}\phi' \, e^{iS_{int}[\phi_c, \phi']}\right] = \mathcal{V}\, V_{eff}(\phi_c)$$

Divide through by spacetime volume factor.

$$V_{eff}(\phi_c) = \frac{1}{2} m^2 \phi_c^2 + \frac{1}{4!}\phi_c^4 + \frac{1}{2}\delta_{m^2}\phi_c^2 + \frac{1}{4!}\delta_\lambda \phi_c^4$$

$$- \frac{i}{2\mathcal{V}} \underbrace{\ln \mathrm{Det}\left(\partial^2 + m_S^2(\phi)\right)}_{\text{One-loop approximation}} + \frac{i}{\mathcal{V}} \underbrace{\ln\left[\int \mathcal{D}\phi' \, e^{iS_{int}[\phi_c, \phi']}\right]}_{\text{Higher order contributions.}}$$

Exercise:

*Coleman-Weinberg effective potential for two degrees of freedom*

Consider a simple field theory describing two interacting real scalar fields, $\phi$ and $\chi$. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - V_{\text{tree}}(\phi,\chi)$$

$$V_{\text{tree}}(\phi,\chi) = -\frac{1}{2}\mu^2\phi^2 + \frac{1}{2}M^2\chi^2 + \frac{a}{4}\phi^2\chi^2 + \frac{\lambda}{4!}\phi^4 + \frac{b}{4!}\chi^4 .$$

where the parameters $\mu^2$, $M^2$, $a$, $b$ and $\lambda$ are all positive. Compute the Coleman-Weinberg effective potential $V_{\text{eff}}(\phi,\chi)$ for $\phi$ and $\chi$ by method of Legendre transform. Make a plot of the effective potential, and compare it to the tree-level potential.

*Hint:* Start by shifting the fields as a change of functional integration variables $\phi(x) \to \bar{\phi} + \phi'(x)$, $\chi(x) \to \bar{\chi} + \chi'(x)$, and identify the terms quadratic in $\phi'$ and $\chi'$. Then perform the functional integral in the gaussian approximation.

$$m^2(\phi,\chi) = \begin{pmatrix} -\mu^2 + \frac{\lambda\bar{\phi}^2}{2} + \frac{a\bar{\chi}^2}{2} & a\bar{\phi}\bar{\chi} \\ a\bar{\phi}\bar{\chi} & M^2 + \frac{a\bar{\phi}^2}{2} + \frac{b\bar{\chi}^2}{2} \end{pmatrix}$$

Then

$$V_{\text{eff}}(\phi,\chi) = V_{\text{tree}}(\phi,\chi) + \frac{1}{64\pi^2}\left[m_1^2(\phi,\chi)\right]^2\left(\ln\frac{m_1^2(\phi,\chi)}{\mu^2} - \frac{3}{2}\right)$$
$$+ \frac{1}{64\pi^2}\left[m_2^2(\phi,\chi)\right]^2\left(\ln\frac{m_2^2(\phi,\chi)}{\mu^2} - \frac{3}{2}\right)$$

where $m^2_{1,2} = \frac{1}{4}\left(2M^2 - 2\mu^2 + (a+\lambda)\phi^2 + (a+b)\chi^2 \pm \sqrt{(\text{stuff})}\right)$

$$\delta_{\Omega} = \frac{1}{16\pi^2}\left(M^4 + \mu^4\right)\left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi\right)$$

$$\delta_{\mu^2} = \frac{1}{16\pi^2}\left(2\lambda\mu^2 - 2aM^2\right)\left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi\right)$$

$$\delta_{M^2} = \frac{1}{16\pi^2}\left(2bM^2 - 2a\mu^2\right)\left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi\right)$$

$$\delta_a = \frac{1}{16\pi^2}\left(8a^2 + 2ab + 2a\lambda\right)\left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi\right)$$

$$\delta_\lambda = \frac{1}{16\pi^2}\left(6a^2 + 6\lambda^2\right)\left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi\right)$$

$$\delta_b = \frac{1}{16\pi^2}\left(6a^2 + 6b^2\right)\left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi\right)$$