Path Integral Approach to Effective Actions (all orders)

Start with the generating functional of the theory, N scalar fields, $i = \{1, \dots N\}$.

$$Z[j] = \int \mathcal{D}\Phi \; e^{iS[\Phi] + i \underbrace{j_x \Phi_x}_{\text{Repeated indices to be integrated over}}} \;, \quad \text{where } S[\Phi] = \int d^4x\, \mathcal{L}[\Phi] \text{ is the action.}$$

Shift the field $\Phi^i(x) \longrightarrow \phi^i_{BG,x}[j] + \phi^i_x$

(as a change of variables in the path integral)

Background field
NOTE: (shift depends on source, j) – explicit dependence given later.

Field configurations that deviate from ϕ_{BG} (to be functionally integrated over.)

– this will give a polynomial in ϕ_{BG} and ϕ in the exponent. Can be split up into 2 parts.

$$S[\phi] \longrightarrow \underbrace{S[\phi_{BG}[j]] + j^i_x \phi^i_{BG,x}[j]}_{\text{A part that only involves the background field, } \phi_{BG}} + \underbrace{\underbrace{j^i_x \cancel{[\phi_{BG}[j]]}\phi^i_x + \tfrac{1}{2}\phi^i_x A^{ij}_{xy} \phi^j_y}_{\text{Part that is linear \& qudratic in } \phi.} + \underbrace{S_{int}[\phi^i_{BG}[j], \phi^i}_{\text{Higher orders in } \phi, \text{ and } \phi - \phi_{BG} \text{ interactio}}}_{= S[\phi_{BG}[j] + \phi] - S[\phi_{BG}[j]] + j^i_x \phi^i_x}$$

and $\mathcal{D}\Phi \to \mathcal{D}\phi$, so that

$$Z[j] = e^{i\left(S[\phi_{BG}[j]] + j^i_x \phi^i_{BG,x}[j]\right)} \int \mathcal{D}\phi \; e^{i\left(S[\phi_{BG}[j] + \phi] - S[\phi_{BG}[j]] + j^i_x \phi^i_x\right)}$$

Recall, $Z[j] = Z[0]\, e^{iW[j]}$

$$(*) \quad W[j] = S[\phi_{BG}[j]] + j^i_x \phi^i_{BG,x}[j] \underbrace{- i \ln \int \mathcal{D}\phi \; e^{i\left(S[\phi_{BG}[j] + \phi] - S[\phi_{BG}[j]] + j^i_x \phi^i_x\right)}}_{\equiv W_{New}[\phi_{BG}[j]]} + i \ln Z[0]$$

$+ i \ln Z[0]$ ← for norm.

$W_{New}[\phi_{BG}[j]]$ ← Need to deal with this.

Since we are after the effective potential, V_{eff}, we need the quantum action, $\Gamma[\phi_c]$. Perform Legendre transformation.

- obtain conjugate field, ϕ_c, to source, j.
- eliminate source, j (and hence $\phi_{BG}[j]$) in favor of ϕ_c.

Field conjugate to source:

$$\phi_c^i(x) = \frac{\delta W[j]}{\delta j^i(x)} = \frac{\delta S[\phi_{BG}]}{\delta \phi^j_{BG,y}} \frac{\delta \phi^j_{BG,y}}{\delta j^i_x} + \phi^i_{BG,x} + j^j_y \frac{\delta \phi^j_{BG,y}}{\delta j^i_x} + \frac{\delta W_{New}[\phi_{BG}]}{\delta \phi^j_{BG,y}} \frac{\delta \phi^j_{BG,y}}{\delta j^i_x}$$

$$= \phi_{BG,x} + \left(\frac{\delta S[\phi_{BG}]}{\delta \phi^j_{BG,y}} + j^j_y + \frac{\delta W_{New}[\phi_{BG}]}{\delta \phi^j_{BG,y}} \right) \frac{\delta \phi^j_{BG,y}}{\delta j^i_x}$$

At this stage, let us specify the functional dependence of $\phi^i_{BG}[j]$ on source, $j^i(x)$: to be such that $(\dots) = 0$.

Hence $\boxed{\phi^i_{BG,x}[j] = \phi^i_{c,x}.}$ $\longleftarrow$ Essentially, we shifted the field, $\Phi^i(x)$, by precisely the amount that makes it conjugate to the source.

$\longrightarrow$ ϕ_{BG} can now trivially be eliminated in favor of ϕ_c. How about $j(x)$? Solve for j^j_y in $(\dots) = 0$ to get:

$$j^j_y = -\frac{\delta S[\phi_{BG}]}{\delta \phi^j_{BG,y}} - \frac{\delta W_{New}[\phi_{BG}]}{\delta \phi^j_{BG,y}},$$

and plug into $W_{new} = -i \ln \int \mathcal{D}\phi \, e^{i S_{New}[\phi_c; \phi]}$

see eqn (*)

$$W_{new}[\phi_{BG}] = -i \ln \int \mathcal{D}\phi \, \exp i \left(S[\phi_{BG} + \phi] - S[\phi_{BG}] - \frac{\delta S[\phi_{BG}]}{\delta \phi^i_{BG,x}} \phi^i_x \underbrace{- \frac{\delta W_{New}[\phi_{BG}]}{\delta \phi^i_{BG,x}}}_{} \phi^i_x \right)$$

Now temporarily define an auxiliary generating functional by replacing this with $\tilde{j}$.

$$W_{aux}[\phi_{BG}; \tilde{j}] = -i \ln \int \mathcal{D}\phi \, \exp i \Bigg(\underbrace{S[\phi_{BG} + \phi] - S[\phi_{BG}] - \frac{\delta S[\phi_{BG}]}{\delta \phi^i_{BG,x}} \phi^i_x}_{\equiv S_{new}[\phi_c; \phi]} + \tilde{j}^i_x \phi^i_x \Bigg).$$

<u>Notice</u>: what does $W_{aux}[\phi_{BG}; \tilde{j}]$ do? It generates connected Green's functions for the theory defined by the new action:

$$S_{New}[\phi_c; \phi] = S[\phi_c + \phi] - S[\phi_c] - \frac{\delta S[\phi_c]}{\delta \phi^i_{c,x}} \phi^i_x$$

$\longleftarrow$ This defines a new theory (whose dynamical field is ϕ), in a background field, ϕ_{BG}.

in our calculations, $\phi_{BG} = \phi_c$.

(**) The special case when $\tilde{j}^{\,i}_{\substack{\text{special}\\\text{case}}}(x) = -\dfrac{\delta W_{New}[\phi_{BG}]}{\delta \phi^i_{BG}(x)}$ gives back $W_{New}[\phi_{BG}]$:

obviously $W_{aux}[\phi_{BG}, j_{\substack{\text{special}\\\text{case}}}] = W_{New}[\phi_{BG}]$.

Recall, that the Legendre transform of W gives the effective action, which generates all 1PI diagrams of the theory. L. transform W_{aux} with respect to $\tilde{j}^i(x)$:

(***) $$\Gamma[\phi_{BG};\tilde{\phi}_c] = W_{aux}[\phi_{BG};\tilde{j}[\tilde{\phi}_c]] - \tilde{j}^{\,i}_x[\tilde{\phi}_c]\,\tilde{\phi}^i_{c,x} \quad \text{with} \quad \tilde{\phi}^i_c = \frac{\delta W_{aux}[\phi_{BG},\tilde{j}]}{\delta \tilde{j}^i(x)}.$$

Furthermore if $\tilde{\phi}_c = 0$, the quantum action is merely the sum of all 1PI vacuum diagrams (apart from a constant), in a background ϕ_{BG}.

n.b. $Z_0 = e^{iW_0}$

$$\Gamma[\phi_{BG}, \tilde{\phi}_c = 0] = W_{aux}[\phi_{BG}; \tilde{j}[\tilde{\phi}_c = 0]] = W^0_{aux} - i\begin{pmatrix}\text{sum of 1PI}\\ \text{vacuum diagrams}\end{pmatrix}$$

For the conjugate variable $\tilde{\phi}^i_c(x)$ to vanish, $\tilde{j}^i(x)$ must have a particular configuration. !! And it turns out that this configuration is precisely the one that gives back W_{New}!

Shown in Jackiw

$$\tilde{j}^i(x) = \frac{-\delta W_{New}[\phi_{BG}]}{\delta \phi^i_{BG}(x)} \quad \text{(see above)}.$$

$$\therefore\ \Gamma[\phi_{BG}, \tilde{\phi}_c = 0] = W_{New}[\phi_{BG}] = W^0_{New}[\phi_{BG}] - i\sum\begin{pmatrix}\text{Vacuum}\\ \text{bubbles }[\phi_{BG}]\end{pmatrix}$$

So, (*) becomes:

$$W[j] = S[\phi_c] + j^i_x[\phi_c]\,\phi^i_{c,x} + W^0_{New}[\phi_c] - i\sum\begin{pmatrix}\text{Vacuum}\\ \text{Bubbles }[\phi_c]\end{pmatrix}$$

Hence, $$\Gamma[\phi_c] = S[\phi_c] + \cancel{j^i_x[\phi_c]\,\phi^i_{c,x}} + W^0_{New}[\phi_c] - i\sum\begin{pmatrix}\text{Vacuum}\\ \text{Bubbles }[\phi_c]\end{pmatrix} - \cancel{j^i_x[\phi_c]\,\phi^i_{c,x}}$$

$$\boxed{\Gamma[\phi_c] = S[\phi_c] + W^0_{New}[\phi_c] - i\sum\begin{pmatrix}\text{Vacuum}\\ \text{Bubbles }[\phi_c]\end{pmatrix} + i\ln Z[0]}$$

n.b. W^0_{New} is obtained by taking the quadratic (free) part of the action in $W_{New}[\phi_c]$ (see bottom of page 2),

$$S_{New}[\phi_c;\phi] = S[\phi_c+\phi] - S[\phi_c] - \frac{\delta S[\phi_c]}{\delta \phi^i_{c,x}}\phi^i_x ,$$

and performing the (gaussian) path integral over ϕ, giving determinants

$$Z_{new}[\phi_c] = \int \mathcal{D}\phi \; e^{i S^{(Quad)}_{New}[\phi_c;\phi]} = \left(\mathrm{Det}[A(\phi_c)]\right)^{-1/2}.$$

then $$W_0 = -i \underbrace{\ln\left[\left(\mathrm{Det}[A(\phi_c)]\right)^{-1/2}\right]}_{\ln Z^{(Quad)}_{New}}.$$

— If there are other fields in the theory, integrate over all of them, obtaing a chain of Det.

We are interested in the effective potential, related to the quantum action by

$$\Gamma[\phi_c] = -V\, V_{eff}(\phi_c) + (\text{derivatives in } \phi)$$

Set $\phi_c(x) \rightarrow \phi_c$ to be independent of spacetime, then

$$V_{eff}(\phi_c) = \underbrace{-\frac{1}{V} S[\phi_c]}_{\text{Tree level}} \underbrace{- \frac{1}{V} W^0_{new}[\phi_c]}_{\substack{\text{One-loop} \\ W^0_{New} = -i \ln Z^{(Quad)}_{New}}} + \underbrace{\frac{i}{V} \sum \left(\begin{matrix}\text{Vacuum} \\ \text{Bubbles } [\phi_c]\end{matrix}\right)}_{\substack{\text{Higher-loop} \\ \text{– defined by } S_{New}[\phi_c;\phi']}}.$$

Since ϕ_c is independent of spacetime label, x, the volume factors should unambiguously factorize from each of the three terms.

$$V_{eff}(\phi_c) = V_0(\phi_c) + \frac{i}{V}\ln Z^{(Quad)}_{New} + \frac{i}{V}\sum\left(\begin{matrix}\text{Vacuum} \\ \text{Bubbles } [\phi_c]\end{matrix}\right)$$

Bosons: $\ln Z^{(Quad)}_{New} \sim \ln\left(\frac{1}{\mathrm{Det}[O]}\right)^{1/2} = -\frac{1}{2}\ln \mathrm{Det}[O]$

Fermions: $\ln Z^{(Quad)}_{new} \sim \ln\left(\mathrm{Det}[O]^{\frac{2}{2} \text{ or } \frac{4}{2}}\right) = (1 \text{ or } 2) \ln \mathrm{Det}[O]$.

1 ≡ Weyl, 2 ≡ Dirac