The Effective Potential — SUMMARY

Start with any gauge theory, gauge group G, not necessarily simple.
 — can be multiple factors, $SU(2) \times U(1)$ in SM.

– Take all the scalars in the theory, and put them in the real representation, $\Phi^i(x)$.

Example: SM: $H = \frac{1}{\sqrt{2}} \begin{pmatrix} G^1 + iG^2 \\ h + iG^3 \end{pmatrix} \longrightarrow \Phi^i(x) = \begin{pmatrix} \Phi^1 \\ \Phi^2 \\ \Phi^3 \\ \Phi^4 \end{pmatrix}$ (no $1/\sqrt{2}$)

S (other scalars) $\longrightarrow (\Phi^5)$

* Be sure to write the generators of the gauge group in the real representation, too.

$$\text{eg:} \quad T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow \frac{1}{2}\left(\begin{array}{cc|cc} & i & & \\ -i & & & \\ \hline & & & -i \\ & & i & \end{array}\right)$$

Pure imaginary.

Actually, it's more convenient to factor out an imaginary unit: $\underset{(\text{imag.})}{T^a} = -i \underset{(\text{Real})}{T^a_{Re}}$ $\Rightarrow$ all generators are purely real from now.

– Put all relevant gauge bosons (gauge eigenstates) in a list, too: $A^a_\mu(x)$.

eg. SM $\quad A^a_\mu(x) = \begin{pmatrix} W^1 \\ W^2 \\ W^3 \\ B \end{pmatrix}$

* if we were looking at a particular SUSY model, and were interested in the effective potential for the colored scalar quarks, then the gluons must be included, too.

Then, the Lagrangian of this theory is merely

$$\mathcal{L} = \frac{1}{2} D_\mu \Phi_i D^\mu \Phi_i - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} - V(\Phi),$$

$$\text{with } (D_\mu \Phi)_i = \partial_\mu \Phi_i + A^a_\mu (\underbrace{g T^a_{Re}}_{\text{The coupling constant associated with generator } a.})_{ij} \Phi_j$$

no i here — absorbed by T^a_{ij}

Next, identify the scalars for which you want the effective potential, and shift them: $\Phi^i(x) = \phi^i(x) + \phi_c^i$

e.g. SM + singlet
$$\begin{pmatrix}\Phi^1\\ \Phi^2\\ \Phi^3\\ \Phi^4\\ \Phi^5\end{pmatrix} = \begin{pmatrix}\phi^1\\ \phi^2\\ \phi^3\\ \phi^4\\ \phi^5\end{pmatrix} + \begin{pmatrix}0\\ 0\\ \phi_c^3\\ 0\\ \phi_c^5\end{pmatrix}$$

Then, the kinetic term, $\frac{1}{2}D_\mu\Phi_i D^\mu\Phi_i = \ldots + \underbrace{A_\mu^a(\partial^\mu\phi_i)}_{} [gT^a\phi_c]_i + \ldots$

Mixing between Longitudinal gauge boson with scalar.

cancel.

— Gauge fix: Add $\mathcal{L}_{GF} = -\frac{1}{2\xi}\left(\partial_\mu A^{\mu a} - \xi [gT^a\phi_c]_i\phi_i\right)^2$

cross term: integrate by parts

Must include compensating ghost terms, $\eta^a(x)$ & $\bar\eta^a(x)$:

$$\mathcal{L}_{Ghost} = \partial_\mu\bar\eta^a\partial^\mu\eta^a + f^{abc}(\partial_\mu\bar\eta^a)\eta^b A^{\mu c} - \xi\bar\eta^a\eta^b(gT^a\phi_c)_i(gT^a\phi_c)_i$$
$$- \xi\bar\eta^a\eta^b(gT^a\phi_c)_i(gT^b_{ij}\phi_j)$$

Adding everything together, (quadratic parts only) — see gauge fixing $[gT^a\phi_c]_i[gT^b\phi_c]_i \equiv m_A^2(\phi_c)^{ab}$
$[gT^a\phi_c]_i[gT^a\phi_c]_j = m_A^2(\phi_c)_{ij}$

$$\mathcal{L}^{Quad} = \frac{1}{2}\phi_i\left(-\partial^2\delta_{ij} - M^2_{ij}(\phi_c) - \xi m_A^2(\phi_c)_{ij}\right)\phi_j$$
$$+ \frac{1}{2}A_\mu^a\left(\partial^2 g^{\mu\nu}\delta^{ab} - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\delta^{ab} + m_A^2(\phi_c)^{ab}g^{\mu\nu}\right)A_\nu^b$$
$$+ \bar\eta^a\left(-\partial^2\delta^{ab} - \xi m_A^2(\phi_c)^{ab}\right)\eta^b + \begin{pmatrix}\text{higher order}\\ \text{in } \phi_i, A, \eta\end{pmatrix}$$

- Since all scalars (physical + goldstone) are together in a single list, the mass matrices $\underbrace{M^2_{ij}(\phi_c) + \xi m_A^2(\phi_c)}_{}$ are also together.

If the fields are shifted properly, the mass matrix for the scalars will look like:

$$\left(\begin{array}{c|c} M^2_{ij}(\phi_c) & 0 \\ \hline 0 & M^2_{ij}(\phi_c)+\xi m_A^2 \end{array}\right) \begin{array}{l} \}\ \text{"physical scalars"} \\ \}\ \text{"goldstones"} \end{array}$$

$$M^2_{ij}(\phi_c) = \left.\frac{\partial^2 V(\Phi)}{\partial\phi_i\partial\phi_j}\right|_{\phi=0}$$

Quantum fluctuations

The Effective potential:

$$V_{\text{eff}}^{\text{1-loop}}(\phi_c) = \frac{i}{V} \ln \Bigg[\underbrace{\int \mathcal{D}\phi \ldots \mathcal{D}A}_{\text{Perform path integral over dynamical fields}} \; e^{\overbrace{i\int d^4x \, \mathcal{L}^{\text{quad}}[\phi_c;\phi,\ldots A]}^{}} \Bigg] - \underbrace{(\phi_c \to 0)}_{\text{"free-field"}}$$

Quadratic part of Action in presence of background field ϕ_c.

Must be determinant-ratio!

Recall: $\int \mathcal{D}\phi \, e^{i\int d^4x \, \frac{1}{2}\phi^i \mathcal{O}^{ij}\phi^j} = \left(\frac{1}{\mathrm{Det}[\mathcal{O}]}\right)^{1/2}$ for bosons.

$$\text{so, } \ln(\text{this}) = -\frac{1}{2}\underbrace{\ln \mathrm{Det}[\mathcal{O}^{ij}]}_{} \leftarrow \text{Functional determinant over indices, } i,j \text{ AND space-time coordinate}$$

$$\equiv \mathrm{Tr} \ln [\mathcal{O}^{ij}]$$

Functional Trace also over indices ij and space-time coordinate

Proceed by diagonalizing the Operators sandwiched between the fields:
$\longrightarrow$ momentum space: $\partial^2 \to -p^2$

$$S[\phi_c;\phi,\ldots A] = \int \frac{d^dp}{(2\pi)^d} \Bigg[\frac{1}{2}\tilde{\phi}_i \left(p^2\delta_{ij} - M^2_{ij}(\phi_c) - \xi m_A^2(\phi_c)_{ij}\right)\tilde{\phi}_j$$

$$+ \frac{1}{2}\tilde{A}^a_\mu \left(\left(-p^2 + m_A^2(\phi_c)^{ab}\right)\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) + \frac{1}{\xi}\left(-p^2 + \xi m_A^2(\phi_c)^{ab}\right)\frac{p^\mu p^\nu}{p^2} \right)\tilde{A}^b_\nu$$

$$+ \tilde{\bar{\eta}}^a \left(p^2\delta^{ab} - \xi m_A^2(\phi_c)^{ab}\right)\eta^b \Bigg]$$

This line written to get in form $(A\,\tilde{P}_T^{\mu\nu} + B\,\tilde{P}_L^{\mu\nu})$.

FINAL STEP: rotate the gauge fields, $\tilde{A}^a_\mu$, by performing a boost, to get $g^{\mu\nu} - p^\mu p^\nu/p^2$ & $p^\mu p^\nu/p^2$ diagonal in the Lorentz indices. $\Lambda^\mu{}_\nu p^\nu \to (|p|, 0$

$$\Lambda^\rho{}_\mu \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)(\Lambda^{-1})_\nu{}^\sigma = \begin{pmatrix} 0 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \ldots \end{pmatrix}^{\rho\sigma} \quad \text{and}$$

$$\Lambda^\rho{}_\mu \left(\frac{p^\mu p^\nu}{p^2}\right)(\Lambda^{-1})_\nu{}^\sigma = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \ldots \end{pmatrix}^{\rho\sigma}.$$

Can't diagonalize mass matrices without specifying a model

$$S[\phi, A, \eta] = \int \frac{d^dp}{(2\pi)^d}\Big[\ \frac{1}{2}\tilde{\phi}_i\left(p^2\delta_{ij} - M^2_{ij}(\phi_c) - \xi m_A^2(\phi_c)_{ij}\right)\tilde{\phi}_j$$

$$+\frac{1}{2}\tilde{A}^a_\mu\left(\left(+p^2 - m_A^2(\phi_c)^{ab}\right)\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots\end{pmatrix}^{\mu\nu} + \frac{1}{\xi}\left(-p^2 + \xi m_A^2(\phi_c)^{ab}\right)\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots\end{pmatrix}^{\mu\nu}\right]\tilde{A}^b_\nu$$

$$+\tilde{\eta}^a\left(p^2\delta^{ab} - \xi m_A^2(\phi_c)^{ab}\right)\tilde{\eta}^b\Big]$$

So, the one-loop effective potential is:

Instructs us to find e-values of $(\ldots)$, take log of each e-value, and sum them.

$$V^{1\text{-loop}}_{\text{eff}} = \frac{-i}{2}\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\Big[\operatorname{Tr}\ln\left(p^2 - M^2_{ij}(\phi_c) - \xi m_A^2(\phi_c)_{ij}\right)$$

$$+(d-1)\operatorname{Tr}\ln\left(p^2 - m_A^2(\phi_c)^{ab}\right) + \operatorname{Tr}\ln\left(\frac{1}{\xi}\left(-p^2 + \xi m_A^2(\phi_c)^{ab}\right)\right)$$

$$-2\operatorname{Tr}\ln\left(p^2 - \xi m_A^2(\phi_c)^{ab}\right)\Big] - \binom{\text{free}}{\text{field}}$$

Factor out $\ln\left(\frac{-1}{\xi}\right)$

Grassman variable (2 DoF) integration.

$$= \frac{-i}{2}\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\Big[\operatorname{Tr}\ln\left(p^2 - M^2_{ij}(\phi_c) - \xi m_A^2(\phi_c)_{ij}\right)$$

scalar (incl. Goldstone) degrees of freedom.

transverse & longitudinal pol. $\quad +(d-1)\operatorname{Tr}\left(p^2 - m_A^2(\phi_c)^{ab}\right)$

Scalar (timelike) pol. of gauge bosons $\quad +\operatorname{Tr}\ln\left(p^2 - \xi m_A^2(\phi_c)^{ab}\right) + \operatorname{Tr}\left(\ln(-1) - \ln(\xi)\right)$

ghosts $\quad -2\operatorname{Tr}\ln\left(p^2 - \xi m_A^2(\phi_c)^{ab}\right)\Big] - \binom{\text{free}}{\text{field}}$

cancel.

$$= \frac{-i}{2}\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\Big[\operatorname{Tr}\ln\left(p^2 - M^2_{ij}(\phi_c) - \xi m_A^2(\phi_c)_{ij}\right)$$

$$+(d-1)\operatorname{Tr}\ln\left(p^2 - m_A^2(\phi_c)^{ab}\right) - \operatorname{Tr}\ln\left(p^2 - \xi m_A^2(\phi_c)^{ab}\right)\Big] - \binom{\text{free}}{\text{field}}$$

(to go to finite temperature, take

$$\int dp^0 \to \frac{i}{\beta}\sum_\omega\ )$$

sum over Matsubara modes.