Verification of the Nielsen Identity at one-loop

Write:
$$V_{\text{eff}} = \underbrace{V^{(0)}}_{\substack{\text{tree-level} \\ \text{eff. potential}}} + \hbar \underbrace{V^{(1)}}_{\substack{\text{one-loop} \\ \text{contribution}}} + \hbar^2 \underbrace{V^{(2)}}_{\substack{\text{two-loop} \\ \text{contribution}}} + \ldots$$

$$C_i(\phi, \xi) = C_i^{(0)} + \hbar C_i^{(1)} + \hbar^2 C_i^{(2)} + \ldots$$

Then, the Nielsen Identity becomes (fields understood to be classical: $\phi_i^c \equiv \phi_i$)

$$\frac{\partial}{\partial \xi}\left(V^{(0)} + \hbar V^{(1)} + \ldots\right) = -\left(C_i^{(0)} + \hbar C_i^{(1)} + \ldots\right)\frac{\partial}{\partial \phi_i}\left(V^{(0)} + \hbar V^{(1)} + \ldots\right)$$

$$= -C_i^{(0)} \frac{\partial V^{(0)}}{\partial \phi_i} - \hbar\left(C_i^{(1)} \frac{\partial V^{(0)}}{\partial \phi_i} + C_i^{(0)} \frac{\partial V^{(1)}}{\partial \phi_i}\right) - \ldots$$

In the generalized R_ξ gauges, the tree level effective potential is gauge invariant. Equating powers of $\hbar$, we have

$$O(\hbar^0): \quad \frac{\partial V^{(0)}}{\partial \xi} = 0 \quad \Rightarrow \quad C_i^{(0)} = 0$$

Hence, to one-loop,

$$O(\hbar): \quad \frac{\partial V^{(1)}}{\partial \xi} = -C_i^{(1)} \frac{\partial V^{(0)}}{\partial \phi_i}$$

__Remember:__ RHS is to be evaluated using ϕ_i^c that is non-vanishing in all its components — i.e. The effective potential must be calculated for all scalars.

Evaluate the LHS — all calculations regularized using Dimensional Regularization ($d = 4 - 2\epsilon$)

Recall:
$$V^{(1)}(\phi) = \frac{-i}{2}\mu^{2\epsilon}\int \frac{d^d p}{(2\pi)^d}\Big[\mathrm{Tr}\ln\left(p^2 - M_{ij}^2(\phi) - \xi m_A^2(\phi)_{ij}\right)$$
$$+ (d-1)\,\mathrm{Tr}\ln\left(p^2 - m_A^2(\phi)^{ab}\right) - \mathrm{Tr}\left(p^2 - \xi m_A^2(\phi)^{ab}\right)\Big]$$

Differentiate with respect to ξ: (suppressing ϕ argument)

$$\text{LHS} = \frac{\partial V^{(1)}}{\partial \xi} = \frac{-i}{2}\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\left[\frac{1}{p^2 - M^2_{ij} - \xi(m_A^2)_{ij}}(-m_A^2)_{ij} - \underbrace{\frac{1}{p^2 - \xi(m_A^2)^{ab}}(-m_A^2)^{ab}}_{=\ \text{Identity: } \frac{1}{p^2-\xi(m_A^2)_{ij}}(-m_A^2)_{ij}}\right]$$

Write as trace:

$$= \frac{-i}{2}\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\,\mathrm{Tr}\Bigg[\underbrace{\frac{1}{p^2 - M^2 - \xi(m_A^2)}}_{\equiv A^{-1}}\underbrace{(-m_A^2)}_{\equiv C} - \underbrace{\frac{1}{p^2 - \xi(m_A^2)}}_{\equiv B^{-1}}\underbrace{(-m_A^2)}_{\equiv C}\Bigg]$$

Combine denominators: Abbr:

$$= \frac{-i}{2}\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\,\mathrm{Tr}[A^{-1}C - B^{-1}C]$$

$$= \frac{-i}{2}\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\,\mathrm{Tr}[\underbrace{B B^{-1} A^{-1} C}_{\text{cyclic}} - B^{-1}A^{-1}AC]$$

$$= \frac{-i}{2}\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\,\mathrm{Tr}[B^{-1}A^{-1}\underbrace{CB}_{[C,B]=0} - B^{-1}A^{-1}AC]$$

$$= \frac{-i}{2}\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\,\mathrm{Tr}[B^{-1}A^{-1}\underbrace{(B-A)}\,C]$$

Use: $B - A = (\cancel{p^2} - \cancel{\xi(m_A^2)}) - (\cancel{p^2} - M^2 - \cancel{\xi(m_A^2)})$
$= M^2$

So, restoring field argument, ϕ, the LHS is:

$$\text{LHS} = \frac{\pm i}{2}\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\,\mathrm{Tr}\left[\frac{1}{p^2 - \xi m_A^2(\phi)}\,\frac{1}{p^2 - M^2(\phi) - \xi m_A^2(\phi)}\,M^2(\phi)\,(\cancel{-}m_A^2(\phi))\right]$$

$$= \frac{+i}{2}\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\,\mathrm{Tr}\left[\frac{1}{p^2 - \xi m_A^2(\phi)}\,\frac{1}{p^2 - M^2(\phi) - \xi m_A^2(\phi)}\,M^2(\phi)\,m_A^2(\phi)\right]$$

Now evaluate RHS. To one-loop, $\mathcal{O}(\hbar)$, only one term contributes to $C_i^{(1)}$:

$$C_i^{(1)} = \frac{-i}{2}\int d^d x \, \langle 0 | T\left(\hat{\bar{\eta}}^a(x)\,(gT^a\phi)_j\,\hat{\phi}(x)\,\hat{\eta}^b(0)\,gT^b_{ik}\,\hat{\phi}(0)\right)|0\rangle_{1PI}$$

The relevant Feynman rules are:

$$a \cdots\cdots\!\blacktriangleright\cdots\cdots b = \langle \hat{\bar{\eta}}^a(x)\,\hat{\eta}^b(0)\rangle = \int \frac{d^4k}{(2\pi)^4}\,\frac{i e^{ik.x}}{k^2 - \xi m_A^2(\phi)^{ab} + i\epsilon}$$

$$j \text{ - - - - - - } k = \langle \hat{\phi}_j(x)\,\hat{\phi}_k(0)\rangle = \int \frac{d^4p}{(2\pi)^4}\,\frac{i e^{ip.x}}{p^2 - M^2_{jk}(\phi) - \xi m_A^2(\phi)_{jk} + i\epsilon}$$

So, the RHS is (suppressing ϕ argument and $+i\epsilon$ prescription):

$$\text{RHS} = -C_i^{(1)}\,\frac{\partial V^{(0)}}{\partial \phi_i}$$

$$= -\frac{-i}{2}\mu^{2\epsilon}\int d^d x \int \frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}\left[\frac{i e^{ik.x}}{k^2 - \xi (m_A^2)^{ab}}\,(gT^a\phi)_j\; gT^b_{ik}\,\frac{i e^{ip.x}}{p^2 - M^2_{jk} - \xi (m_A^2)_{jk}}\,\frac{\partial V^{(0)}}{\partial\phi_i}\right]$$

Integrate over x obtaining δ-function: $(2\pi)^d\,\delta^d(k+p)$.
Integrate over k fixing $k \to -p$.
Collect indicated factors of i to front.

$$= \underbrace{-\frac{-i}{2}\,i^2}_{-i/2}\,\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\left[\underbrace{\frac{1}{p^2 - \xi (m_A^2)^{ab}}\,(gT^a\phi)_j}_{①}\; \underbrace{gT^b_{ik}\,\frac{1}{p^2 - M^2_{jk} - \xi(m_A^2)_{jk}}\,\frac{\partial V^{(0)}}{\partial\phi_i}}_{②}\right]$$

Use Identities:

① $$\frac{1}{p^2 - \xi (m_A^2)^{ab}}\,(gT^a\phi)_j = \frac{1}{p^2 - \xi (m_A^2)_{\ell j}}\,(gT^b\phi)_\ell$$

② $$gT^b_{ik}\,\frac{\partial V^{(0)}}{\partial\phi_i} = -(gT^b\phi)_i\,\frac{\partial^2 V^{(0)}}{\partial\phi_i\,\partial\phi_k} = (-)(gT^b\phi)_i\,M^2_{ik}$$

cancels minus sign in front

minus sign cancelled from identity ②.

$$\text{RHS} = \frac{i}{2}\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\left[\frac{1}{p^2-\xi(m_A^2)_{\ell j}}\underbrace{(gT^b\phi)_\ell(gT^b\phi)_i}_{=m_A^2(\phi)_{\ell i}}\frac{1}{p^2-M^2_{jk}-\xi(m_A^2)_{jk}}M^2_{ik}\right]$$

symmetric

$$= \frac{i}{2}\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\left[\frac{1}{p^2-\xi(m_A^2)_{\ell j}}\frac{1}{p^2-M^2_{jk}-\xi(m_A^2)_{jk}}M^2_{ki}(m_A^2)_{i\ell}\right]$$

This is in matrix multiplication order — write as trace (restoring ϕ argument)

$$\text{RHS} = \frac{i}{2}\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\,\text{Tr}\left[\frac{1}{p^2-\xi m_A^2(\phi)}\frac{1}{p^2-M^2(\phi)-\xi m_A^2(\phi)}M^2(\phi)\,m_A^2(\phi)\right]$$

In agreement with LHS. $\Rightarrow$ Nielsen identity verified to one-loop.

<u>NOTICE</u>: When evaluated at the tree-level minimum, $\phi_c = v_0$, we have $M^2(v_0)\,m_A^2(v_0) = 0$. $\Rightarrow$ RHS = LHS = 0 as expected.

Can readily evaluate RHS in scalar QED since mass matrices are already diagonal:

$$m_A^2(\phi_c)_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & e^2\phi_c^2 \end{pmatrix} \qquad M_{ij}^2(\phi_c) = \begin{pmatrix} m^2+3\lambda\phi_c^2 & \\ & m^2+\lambda\phi_c^2 \end{pmatrix}$$

The propagators are:

$$\frac{1}{p^2 - \xi m_A^2(\phi_c)} = \begin{pmatrix} \frac{1}{p^2} & \\ & \frac{1}{p^2-\xi e^2\phi_c^2} \end{pmatrix}$$

$$\frac{1}{p^2 - M^2(\phi_c) - \xi m_A^2(\phi_c)} = \begin{pmatrix} \frac{1}{p^2-m^2-3\lambda\phi_c^2} & \\ & \frac{1}{p^2-m^2-\lambda\phi_c^2-\xi e^2\phi_c^2} \end{pmatrix}$$

So, to one-loop

$$\frac{\partial V}{\partial \xi} = \frac{i}{2}\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\,\mathrm{Tr}\left[\begin{pmatrix} \frac{1}{p^2} & \\ & \frac{1}{p^2-\xi e^2\phi_c^2} \end{pmatrix}\begin{pmatrix} \frac{1}{p^2-m^2-3\lambda\phi_c^2} & \\ & \frac{1}{p^2-m^2-\lambda\phi_c^2-\xi e^2\phi_c^2} \end{pmatrix}\right.$$
$$\left.\times\begin{pmatrix} m^2+3\lambda\phi_c^2 & \\ & m^2+\lambda\phi_c^2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & e^2\phi_c^2 \end{pmatrix}\right]$$

$$= \frac{i}{2}(m^2+\lambda\phi_c^2)e^2\phi_c^2\;\mu^{2\epsilon}\int\frac{d^dp}{(2\pi)^d}\,\frac{1}{p^2-\xi e^2\phi_c^2}\,\frac{1}{p^2-m^2-\lambda\phi_c^2-\xi e^2\phi_c^2}$$

(Standard one-loop integral)

$$\frac{\partial V}{\partial \xi} = \frac{i}{2}(m^2+\lambda\phi_c^2)e^2\phi_c^2\;\frac{i}{(4\pi)^2}\left\{\frac{1}{\epsilon} - \gamma_E + \ln 4\pi\right.$$
$$-\frac{1}{m^2+\lambda\phi_c^2+\xi e^2\phi_c^2}\left[m^2+\lambda\phi_c^2 - (m^2+\lambda\phi_c^2+\xi e^2\phi_c^2)\ln\left(\frac{m^2+\lambda\phi_c^2+\xi e^2\phi_c^2}{\mu_R^2}\right)\right.$$
$$\left.\left.+\,\xi e^2\phi_c^2\ln\left(\frac{\xi e^2\phi_c^2}{\mu_R^2}\right)\right]\right\}$$

Note that $\xi \to 0$ limit (Landau gauge) is well-behaved, and non-zero.