Gauge Dependence of the Effective Action (for ONLY scalors)
$A^\mu_{cl} \to 0$

Derivative expansion (in ∂_μ):

$$\Gamma[\phi] = \int d^4x \left[V_{eff}(\phi) + \frac{1}{2} Z_{ij}(\phi)\, \partial_\mu \phi_i \partial^\mu \phi_j + \ldots \right]$$

Each term is expanded in $\hbar$ (loop-expansion)

$$\Gamma[\phi] = \int d^4x \Big[V^{(0)}(\phi) + \hbar V^{(1)}(\phi) + \ldots + \frac{1}{2}\big(\underbrace{Z^{(0)}_{ij}(\phi)}_{=\delta_{ij}\ \text{(canonical normalization)}} + \hbar Z^{(1)}_{ij}(\phi) + \ldots \big) \partial_\mu \phi_i \partial^\mu \phi_j + \ldots \Big]$$

$$= \int d^4x \left[V^{(0)}(\phi) + \hbar^{(1)} V^{(1)}(\phi) + \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i + \frac{1}{2} \hbar Z^{(1)}_{ij}(\phi) \partial_\mu \phi_i \partial^\mu \phi_j + \ldots \right.$$

The coefficient of each term is, in principle, gauge (ξ) dependent

CONTEXT: Bubble Nucleation Rate

$$\begin{pmatrix}\text{Nucleation}\\ \text{Rate}\end{pmatrix} = \begin{pmatrix}\text{Pre-exponential}\\ \text{factor}\end{pmatrix} e^{-\Gamma_E[\phi_B]}$$

with ϕ_B satisfying $\frac{\delta \Gamma}{\delta \phi_B} = 0$ "Bounce solution"

Is the nucleation rate ξ independent?
- Answer: Use Nielsen's identities.

Nielsen Identity

$$\frac{\delta \Gamma}{\delta \xi} = \int d^4x \left(\int d^4y \frac{\delta^2 \Gamma}{\delta K_{GF}(y)\, \delta (k^c_\phi)_k(x)} \right) \frac{\delta \Gamma}{\delta \phi_k(x)} \quad \text{(sum over } k\text{)}$$

Clearly, if Γ & ϕ_B were found exactly to all orders in $\hbar$ & ∂_μ, the Nuc. rate will be ξ indep since $\frac{\delta \Gamma}{\delta \phi} = 0$

But at, finite order, LHS of Nielsen's identity is a double series in $\hbar$ and ∂_μ. $\Rightarrow$ RHS must also be a double series.

Write: $$\int d^4y \,\frac{\delta^2\Gamma}{\delta K_{GF}(y)\,\delta(K_\phi^c)_k(x)} = -\left(C_k(\phi,\xi) + D_k^{ij}(\phi,\xi)\,\partial_\mu\phi_i\,\partial^\mu\phi_j + \dots\right)$$

$$= -\left[C_k^{(0)} + \hbar\, C_k^{(1)} + \dots + \left(D_k^{ij\,(0)} + \hbar\, D_k^{ij\,(1)} + \dots\right)\partial_\mu\phi_i\,\partial^\mu\phi_j + \dots\right]$$

and

$$\frac{\delta\Gamma}{\delta\phi_k(x)} = \frac{\partial V^{(0)}}{\partial\phi_k} + \hbar\,\frac{\partial V^{(1)}}{\partial\phi_k} + \dots - \partial_\mu\partial^\mu\phi_k + \hbar\left(\frac{1}{2}\frac{\partial Z_{ij}^{(1)}}{\partial\phi_k}\partial_\mu\phi_i\partial^\mu\phi_j - \partial_\mu\left(Z_{ij}^{(1)}(\phi)\,\partial^\mu\phi_k\right)\right) + \dots$$

So, the Nielsen Identity becomes:

$$\frac{\partial\Gamma}{\partial\xi} = \int d^4x\left[\frac{\partial V^{(0)}}{\partial\xi} + \hbar\,\frac{\partial V^{(1)}}{\partial\xi} + \hbar\,\frac{1}{2}\frac{\partial Z_{ij}^{(1)}}{\partial\xi}\partial_\mu\phi_i\partial^\mu\phi_j + \dots\right] \quad \leftarrow \text{LHS}$$

$$= \int d^4x\left[C_k^{(0)}\frac{\partial V^{(0)}}{\partial\phi_k} + \hbar\left(C_k^{(0)}\frac{\partial V^{(1)}}{\partial\phi_k} + C_k^{(1)}\frac{\partial V^{(0)}}{\partial\phi_k}\right)\right. \quad \leftarrow \text{RHS}$$

$$+\left(-C_k^{(0)}\,\partial_\mu\partial^\mu\phi_k + D_k^{ij\,(0)}\frac{\partial V^{(0)}}{\partial\phi_k}\partial_\mu\phi_i\partial^\mu\phi_k\right)$$

$$+\hbar\left(C_k^{(0)}\left[\frac{1}{2}\frac{\partial Z_{ij}^{(1)}}{\partial\phi_k}\partial_\mu\phi_i\partial^\mu\phi_j - \partial_\mu\left(Z_{ij}^{(1)}(\phi)\,\partial^\mu\phi_k\right)\right] - C_k^{(1)}\partial_\mu\partial^\mu\phi_k + D_k^{ij\,(0)}\frac{\partial V^{(1)}}{\partial\phi_k}\partial_\mu\phi_i\partial^\mu\phi_j\right.$$

$$\left.\left. + D_k^{ij\,(1)}\frac{\partial V^{(0)}}{\partial\phi_k}\partial_\mu\phi_i\partial^\mu\phi_j\right)\right]$$

Match terms order-by-order in $\hbar$ & ∂_μ to obtain C & D coeffs:

$\mathcal{O}(\hbar^0, \partial_\mu^0)$: Since tree-level potential is ξ-indep in R_ξ gauges,

$$\frac{\partial V^{(0)}}{\partial\xi} = 0 \quad\Rightarrow\quad C_k^{(0)} = 0.$$

$\mathcal{O}(\hbar^0, \partial_\mu^1)$: $$\underbrace{0}_{\substack{\text{No term}\\ \text{in LHS}}} = -\underbrace{C_k^{(0)}}_{\substack{=0\\ \text{(just established}\\ \text{above)}}}\partial_\mu\partial^\mu\phi_k + D_k^{ij\,(0)}\frac{\partial V^{(0)}}{\partial\phi_k}\partial_\mu\phi_i\partial^\mu\phi_k$$

In, general this is non-vanishing

$$\Rightarrow D_k^{ij\,(0)} = 0.$$

So then,

$$\frac{\partial \Gamma}{\partial \xi} = \int d^4x \left[\hbar\, C_k^{(1)} \frac{\partial V^{(0)}}{\partial \phi_k} - \hbar\, C_k^{(1)} \partial_\mu \partial^\mu \phi_k + \hbar\, D_k^{ij\,(1)} \frac{\partial V^{(0)}}{\partial \phi_k} \partial_\mu \phi_i \partial^\mu \phi_j \right]$$

$$= \hbar \int d^4x \left[C_k^{(1)} \left(\frac{\partial V^{(0)}}{\partial \phi_k} - \Box\, \phi_k \right) + D_k^{ij\,(1)} \frac{\partial V^{(0)}}{\partial \phi_k} \partial_\mu \phi_i \partial^\mu \phi_j \right]$$

If the bounce solution is constructed from the tree-level equation of motion: $\partial V^{(0)}/\partial \phi - \Box\, \phi = 0$, then the 1^{st} term vanishes.

If, in addition, it can be arranged for $D_k^{ij\,(1)}$ to vanish [see Metaxas and Weinberg $\leftarrow$ they drop $\lambda \sim \mathcal{O}(e^4)$ contribution to $Z(\phi)$], the 2^{nd} term also vanishes $\Rightarrow$ the tunneling rate becomes gauge independent.

OTHERWISE, the tunneling rate will be gauge dependent at this order in perturbation theory.