Gauge independence of sgn(curvature)

$$\text{Curvature} \equiv k_{ij} = \left.\frac{\partial^2 V(\xi)}{\partial\phi_i \partial\phi_j}\right|_{\phi^{(1)}(\xi)}$$

Differentiate :

$$\frac{dk_{ij}}{d\xi} = \left.\frac{\partial^3 V}{\partial\phi_i \partial\phi_j \partial\phi_k}\right|_{\phi^{(1)}} \underbrace{\frac{\partial\phi_k^{(1)}}{\partial\xi}}_{= C_k} + \left.\frac{\partial^3 V}{\partial\phi_i \partial\phi_j \partial\xi}\right|_{\phi^{(1)}}$$

$= C_k$ (see prev. pg)

Use N. id on this.

$$\frac{\partial V}{\partial\xi} = -C_k \frac{\partial V}{\partial\phi_k}$$

$$\Rightarrow \frac{\partial^2 V}{\partial\phi_j \partial\xi} = -\frac{\partial C_k}{\partial\phi_j}\frac{\partial V}{\partial\phi_k} - C_k \frac{\partial^2 V}{\partial\phi_j \partial\phi_k}$$

$$\Rightarrow \frac{\partial^3 V}{\partial\phi_i \partial\phi_j \partial\xi} = -\frac{\partial^2 C_k}{\partial\phi_i \partial\phi_k}\frac{\partial V}{\partial\phi_k} - \frac{\partial C_k}{\partial\phi_j}\frac{\partial^2 V}{\partial\phi_i \partial\phi_k} - \frac{\partial C_k}{\partial\phi_i}\frac{\partial^2 V}{\partial\phi_j \partial\phi_k} - C_k \frac{\partial^3 V}{\partial\phi_i \partial\phi_j \partial\phi_k}$$

Evaluate at $\phi^{(1)} \Rightarrow 1^{st}$ term vanishes $\partial V/\partial\phi = 0$.

$$\therefore \frac{dk_{ij}}{d\xi} = \cancel{\left.\frac{\partial^3 V}{\partial\phi_i \partial\phi_j \partial\phi_k}\right|_{\phi^{(1)}} C_k} - \left.\left(\frac{\partial C_k}{\partial\phi_j}\frac{\partial^2 V}{\partial\phi_i \partial\phi_k} + \frac{\partial C_k}{\partial\phi_i}\frac{\partial^2 V}{\partial\phi_j \partial\phi_k}\right)\right|_{\phi^{(1)}} \cancel{- C_k \left.\frac{\partial^3 V}{\partial\phi_i \partial\phi_j \partial\phi_k}\right|_{\phi^{(1)}}}$$

$$\frac{dk_{ij}}{d\xi} = -\frac{\partial C_k}{\partial\phi_i} k_{kj} - k_{ik}\frac{\partial C_k}{\partial\phi_j}$$