Gelfand-Yaglom Formula

Suppose want to compute determinant arising from:

$$F(t_f, t_i) = \int_{t_i}^{t_f} dt \left[\frac{1}{2} m \delta\dot{x}^2 - \frac{1}{2} m \Omega^2(t)\, \delta x^2 \right] \qquad \Omega^2(t) \equiv \text{arbitrary function of } t.$$

$$= \sqrt{\frac{m}{2\pi i\hbar(t_f - t_i)}} \left(\frac{\det\left[-\frac{\partial^2}{\partial t^2} - \Omega^2(t) \right]}{\det\left[-\frac{\partial^2}{\partial t^2} \right]} \right)^{-1/2}$$

Go back to time-sliced notation,

$$= \sqrt{\frac{m}{2\pi i\hbar(t_f - t_i)}} \left(\frac{\det\left[\delta t^2 \left(-\nabla\bar{\nabla} - \Omega^2(t) \right) \right]}{\det\left[\delta t^2 \left(-\nabla\bar{\nabla} \right) \right]} \right)^{-1/2},$$

and write:

$$\Omega^2(t) = \begin{pmatrix} \Omega_1^2 & & & \\ & \Omega_2^2 & & \\ & & \ddots & \\ & & & \Omega_N^2 \end{pmatrix}.$$

Can get recursive formula for determinant of $\delta t^2 \left(-\nabla\bar{\nabla} - \Omega^2(t) \right)$:

Define:

$\mathbb{D}_{N-1}$ $\qquad$ $\mathbb{D}_N$

$$D_N \equiv \det \mathbb{D}_N = \begin{pmatrix} 2 - \delta t^2 \Omega_1^2 & -1 & & & & \\ -1 & 2 - \delta t^2 \Omega_2^2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 - \delta t^2 \Omega_{N-2}^2 & -1 & \\ & & & -1 & 2 - \delta t^2 \Omega_{N-1}^2 & -1 \\ & & & & -1 & 2 - \delta t^2 \Omega_N^2 \end{pmatrix}$$

Using Laplace expansion (in terms of cofactors) on last column:

$$\begin{pmatrix} \mathbb{D}_{N-2} & -1 \\ 0 & -1 \end{pmatrix}$$

$$\det \mathbb{D}_N = \left(2 - \delta t^2 \Omega_N^2 \right) \det \mathbb{D}_{N-1} - (-1) \underbrace{\det \tilde{\mathbb{D}}_{N-2}}_{(-1)\det \mathbb{D}_{N-2}}$$

compute by L's expansion in last row

or,

$$D_N = \left(2 - \delta t^2 \Omega_N^2 \right) D_{N-1} - D_{N-2} \qquad \left(D_N \equiv \det \mathbb{D}_N \right)$$

to solve for D_N.

Rearrange.

$$D_N = 2D_{N-1} - \delta t^2\,\Omega_N^2\,D_{N-1} - D_{N-2}$$

$$0 = (D_N - D_{N-1}) - (D_{N-1} - D_{N-2}) + \delta t^2\,\Omega_N^2\,D_{N-1}$$

$$= \delta t^2\left[\frac{1}{\delta t}\left(\frac{D_N - D_{N-1}}{\delta t} - \frac{D_{N-1} - D_{N-2}}{\delta t}\right) + \Omega_N^2\,D_{N-1}\right]$$

Can view $D(t) \equiv \{D_1, D_2, \ldots D_N\}$ as a function of t, formed by collecting the sequence of determinants and making a list.

$$0 = \underbrace{\cancel{\delta t^2}}_{\text{divide out}}\left[\frac{1}{\delta t}\left(\bar{\nabla} D_{N-1} - \bar{\nabla} D_{N-2}\right) + \Omega_N^2\,D_{N-1}\right]$$

$$0 = \nabla\bar{\nabla}\,D_{N-1} + \Omega_N^2\,D_{N-1} \qquad \text{or} \qquad \left(\nabla\bar{\nabla} + \Omega_N^2\right) D_{N-1} = 0$$

Valid for all N $\Longrightarrow$ (not necessarily "the last entry") relabel $N \to i+1$

$$\boxed{\left(\nabla\bar{\nabla} + \Omega_{i+1}^2\right) D_i = 0} \qquad \text{or} \qquad \left[(\nabla\bar{\nabla})_{ij} + \begin{pmatrix}\Omega_2 & & & \\ & \Omega_3 & & \\ & & \ddots & \\ & & & \Omega_{N+1}\end{pmatrix}_{ij}\right]\begin{pmatrix}D_1\\ D_2\\ \vdots \\ D_N\end{pmatrix}_j$$

<u>Gelfand-Yaglom Formula</u>

Initial conditions ① $D_1 \equiv \det \mathbb{D}_1 = 2 - \delta t^2\,\Omega_1^2$

② $D_2 \equiv \det \mathbb{D}_2 = (2 - \delta t^2\,\Omega_1^2)(2 - \delta t^2\,\Omega_2^2) - 1$

$D_N \equiv \det(\mathbb{D}_N)$ is the answer we seek

Continuum Limit:

Solve differential eqn: $\left(\dfrac{\partial^2}{\partial t^2} + \Omega^2(t)\right)\left[\overset{\text{multiplied by } \delta t \text{ for convenience}}{\delta t\ D(t)}\right] = 0$

subject to initial conditions.

① $\displaystyle\lim_{\delta t\to 0}\left[\delta t\ D_1\right] = \delta t\, D(t_i) = 0$

AND

② $\displaystyle\lim_{\delta t\to 0}\frac{1}{\delta t}\left[\delta t\, D_2 - \delta t\, D_1\right] = \delta t\,\dot{D}(t_i) = (2 - \delta t^2\Omega_1^2)(2 - \delta t^2\Omega_2^2) - 1 - (2 - \delta t^2\Omega_1^2)$

$= 4 - 1 - 2 = 1$

then $\det\left(-\dfrac{\partial^2}{\partial t^2} - \Omega^2(t)\right) = \dfrac{1}{\delta t}\left[\delta t\ D(t_f)\right]$

Example: Gelfand–Yaglom formula applied to harmonic oscillator:

Fluctuation factor:

$$F = \sqrt{\frac{m}{2\pi i\hbar(t_f - t_i)}} \left(\frac{\det\left[-\frac{\partial^2}{\partial t^2} - \omega^2\right]}{\det\left[-\frac{\partial^2}{\partial t^2}\right]} \right)^{-1/2}$$

NUMERATOR — solve diff eq:

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right) [\delta t\, D(t)] = 0 \,, \qquad \text{B.C.:} \quad \delta t\, D(t_i) = 0, \quad \delta t\, \dot{D}(t_f) = 1$$

solution: $\delta t\, D(t) = \frac{1}{\omega} \sin(\omega(t - t_i)) \Rightarrow \det\left[-\frac{\partial^2}{\partial t^2} - \omega^2\right] = \frac{1}{\delta t\, \omega} \sin\omega(t_f - t_i)$

DENOMINATOR — solve diff eq:

$$\frac{\partial^2}{\partial t^2} [\delta t\, D(t)] = 0 \qquad (\text{same B.C.})$$

Solution: $\delta t\, D(t) = (t - t_i) \quad \Rightarrow \quad \det\left[-\frac{\partial^2}{\partial t^2}\right] = \frac{1}{\delta t} (t_f - t_i)$

Hence, the fluctuation factor is:

$$F = \sqrt{\frac{m}{2\pi i\hbar\cancel{(t_f - t_i)}}} \left(\frac{\frac{\cancel{1}}{\cancel{\delta t}\,\omega} \sin\omega(t_f - t_i)}{\frac{\cancel{1}}{\cancel{\delta t}} \cancel{(t_f - t_i)}} \right)^{-1/2}$$

$$= \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega(t_f - t_i)}} \,, \qquad \text{in agreement with e-value method.}$$