Gaussian Functional Integral

1D Gaussian Integral: $I(a) = \int_{-\infty}^{+\infty} d\phi \; e^{-\frac{1}{2}a\phi^2} = \sqrt{\frac{2\pi}{a}}$

Generalize to N degrees of freedom (real variables $\phi_1, \dots, \phi_N$)

$$I(A) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} d\phi_1 \cdots d\phi_N \; e^{-\frac{1}{2}\sum_i \sum_j \phi_i A_{ij} \phi_j}$$

Exponent is in matrix form: $\frac{1}{2}\phi_i A_{ij} \phi_j$ — take A_{ij} to be symmetric.
Real symmetric matrix A can be diagonalized by an orthogonal matrix R.

ie. $A = R^T D R$

$$\longrightarrow \; -\frac{1}{2}\sum_i \sum_j \underbrace{\phi_i R_{ij}}_{y_j} D_{jk} \underbrace{R_{kl} \phi_l}_{y_k} \qquad y_j = \phi_i R_{ij}$$

$$I(A) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} dy_1 \cdots dy_N \; e^{-\frac{1}{2} y^T D y}$$

$$= \left(\int_{-\infty}^{+\infty} dy_1 \; e^{-\frac{1}{2}y_1^2 d_1}\right) \cdots \left(\int_{-\infty}^{+\infty} dy_N \; e^{-\frac{1}{2}y_N^2 d_N}\right)$$

$$= I(d_1) \cdots I(d_N)$$

$$= (2\pi)^{N/2} (d_1 d_2 \dots d_N)^{-1/2}$$

But $d_1 \dots d_N$ = product of e-values = $\det D = \det A$

$$\therefore \; I(A) = \left(\frac{(2\pi)^N}{\det(A)}\right)^{1/2}$$

Continuum Limit: define: $\int_{-\infty}^{+\infty} \sqrt{\frac{1}{2\pi}}\, d\phi_1 \cdots \int_{-\infty}^{+\infty} \sqrt{\frac{1}{2\pi}}\, d\phi_n = \int \mathcal{D}\phi$

Note that the $\sqrt{1/2\pi}$ are in here.

$$\int \mathcal{D}\phi \; e^{-\frac{1}{2}\int dx \int dy \; \phi(x) A(x,y) \phi(y)} = \int \mathcal{D}\phi \; e^{-S_0[\phi]}$$

for $\int \mathcal{D}\phi \; e^{i\int dx \int dy \; \frac{1}{2}\phi(x) A(x,y) \phi(y)} = \int \mathcal{D}\phi \; e^{iS_0[\phi]}$,

$\mathcal{D}\phi$ contains $\sqrt{\frac{1}{2\pi i}} \cdots \sqrt{\frac{1}{2\pi i}}$

NOTE!!

Normalization of $\mathcal{D}\phi$ is context dependent!

Gaussian Functional Integral

1D complex integral: $$I(a) = \underbrace{\int d\phi\, d\phi^*}_{\substack{\text{understood to mean} \\ \text{integration over all complex } \phi \text{ plane.}}} e^{-a\phi^*\phi}$$

$$\equiv \int_{-\infty}^{+\infty} d(\mathrm{Re}\,\phi) \int_{-\infty}^{+\infty} d(\mathrm{Im}\,\phi)$$

Move to polar representation:

$$I(a) = \underbrace{\int_0^{2\pi} d\phi}\, \int_0^{\infty} d|\phi|\; |\phi|\, e^{-a|\phi|^2}$$

$$= \cancel{2}\pi \times \frac{1}{\cancel{2}a} = \frac{\pi}{a}$$

Generalize to N degrees of freedom:

$$I(A) = \int d\phi_1 d\phi_1^* \; d\phi_2 d\phi_2^* \ldots d\phi_N d\phi_N^* \; e^{-\sum_i \sum_j \phi_i^* A_{ij} \phi_j}$$

Take A to be (positive-definite) hermitian matrix. Can be diagonalized by a unitary matrix U. i.e. $A = U^\dagger \mathbb{D}\, U$ ← diagonal.

ch. var. $\phi_i = U_{ij}\, y_j$ & $\phi_i^* = U^\dagger_{ij}\, \bar{y}_j$

$$I(A) = \int dy_1 d\bar{y}_1 \cdots dy_N d\bar{y}_N \; e^{-\bar{y}\mathbb{D}y}$$

$$= \left(\int dy_1 d\bar{y}_1 \; e^{-\bar{y}_1 d_1 y_1}\right) \cdots \left(\int dy_N d\bar{y}_N \; e^{-\bar{y}_N d_N y_N}\right)$$

$$= I(d_1) \ldots I(d_N)$$

$$= \pi^N \underbrace{(d_1 d_2 \ldots d_N)}_{\text{Product of e-values}}{}^{-1}$$

Product of e-values $\equiv \det \mathbb{D} = \det A \quad \Rightarrow \quad I(A) = \dfrac{\pi^N}{\mathrm{Det}\, A}$

Continuum limit:

define: $$\int \frac{1}{\pi} d\phi_1 d\phi_1^* \ldots \int \frac{1}{\pi} d\phi_N d\phi_N^* \equiv \int \mathcal{D}\phi\, \mathcal{D}\phi^*$$

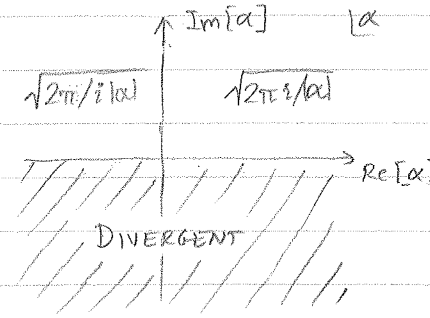
$$\int \mathcal{D}\phi\, \mathcal{D}\phi^* \; e^{-\int d^4x\; \phi^\dagger(-\partial^2 - m^2)\phi} = Z.$$

NOTE! normalization of $\mathcal{D}\phi$ is context dependent.

<u>Fresnel's Formula:</u>

$$I(\alpha) = \int_{-\infty}^{+\infty} dx\, e^{i\alpha x^2/2} = \sqrt{\frac{2\pi}{|\alpha|}} \times \begin{cases} \sqrt{i}, & \mathrm{Re}[\alpha] > 0 \\ 1/\sqrt{i}, & \mathrm{Re}[\alpha] < 0 \end{cases}$$

$(\mathrm{Im}\,\alpha > 0)$



derived from standard Gaussian integral (and setting $a = -i\alpha$)

$$I_G(a) = \int_{-\infty}^{\infty} dx\; e^{-ax^2/2} = \sqrt{2\pi/a}$$

$(\mathrm{Re}\, a > 0)$