Quantum Time Correlation Functions (Path Integrals)

Start with

$$\int \mathcal{D}\phi \; \phi(x_1)\,\phi(x_2)\; e^{iS} = \int \mathcal{D}\phi \; \phi(t_1, \vec{x}_1)\,\phi(t_2, \vec{x}_2)\; e^{iS}$$

Split integration measure into three parts:

$$= \int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \underbrace{\int_{\substack{\phi(t_1,\vec{x})=\phi_1(\vec{x}) \\ \phi(t_2,\vec{x})=\phi_2(\vec{x})}} \mathcal{D}\phi}_{} \quad \phi(t_1, \vec{x}_1)\,\phi(t_2, \vec{x}_2)\; e^{iS}$$

$\phi(t_2,\vec{x})=\phi_2(\vec{x}) \longleftarrow$ smooth B.C.'s

Pull $\phi(t_1, \vec{x}_1)$ and $\phi(t_2, \vec{x}_2)$ outside the integral, renaming them $\phi_1(\vec{x})$ & $\phi_2(\vec{x})$:

$$= \underbrace{\int \mathcal{D}\phi_1(\vec{x}) \;\; \phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \;\; \phi_2(\vec{x})} \; \underbrace{\int_{\substack{\phi(t_1,\vec{x})=\phi_1(\vec{x}) \\ \phi(t_2,\vec{x})=\phi_2(\vec{x})}} \mathcal{D}\phi \quad e^{iS}}$$

This is a three-functional integral (as opposed to a four-functional integral)

Interpretation of this is known:
It is the amplitude of system going from $\phi_{init} \to \phi_1$, and $\phi_1 \to \phi_2$ and finally $\phi_2 \to \phi_{final}$ (if $t_1 < t_2$).
Otherwise, it is $\phi_{init} \to \phi_2$; $\phi_2 \to \phi_1$; $\phi_1 \to \phi_{final}$ if $t_2 < t$

Suppose $t_1 < t_2$, i.e t_1 earlier than t_2.

$$= \int \mathcal{D}\phi(\vec{x}_1)\, \phi(\vec{x}_1) \int \mathcal{D}\phi_2(\vec{x})\, \phi_2(\vec{x}) \;\; \langle \phi_{final} | \, e^{-i\hat{H}(T - t_2)} | \phi_2 \rangle \langle \phi_2 | e^{-i\hat{H}(t_2 - t_1)} | \phi_1 \rangle$$

$$\times \langle \phi_1 | \, e^{-i\hat{H}(t_1 + T)} \, | \phi_{init} \rangle$$

$T \equiv$ large past and future time.

Since $|\phi_1\rangle$ and $|\phi_2\rangle$ are states of definite field configuration, they are e-states of $\hat{\phi}_1$ and $\hat{\phi}_2$: $\quad \hat{\phi}(\vec{x})\, |\Phi\rangle = \Phi(\vec{x})\, |\phi_1\rangle$.

$$= \langle \phi_{final} | \, e^{-i\hat{H}(T-t_2)} \, \hat{\phi}(\vec{x}_2) \underbrace{\int \mathcal{D}\phi_2 \, |\phi_2\rangle\langle\phi_2|}_{=1} \, e^{-i\hat{H}(t_2 - t_1)} \, \hat{\phi}(\vec{x}_1) \underbrace{\int \mathcal{D}\phi_1 |\phi_1\rangle\langle\phi_1|}_{=1} \, e^{-i\hat{H}(t_1+T)} \, |\phi_{init}\rangle$$

(completeness relation)

So,

$$\int \mathcal{D}\phi\ \phi(x_1)\,\phi(x_2)\,e^{iS} = \langle \phi_{\text{final}} |\, e^{-i\hat{H}(T-t_2)}\ \hat{\phi}(\vec{x}_2)\ e^{-i\hat{H}(t_2-t_1)}\ \hat{\phi}(\vec{x}_1)\ e^{-i\hat{H}(t_1+T)}\, | \phi_{\text{init}} \rangle$$

$(t_1 < t_2)$

These operators are in the Schrödinger picture (are indep. of time)

Combine exponentials with field operators to express them in terms of time-dependent Heisenberg picture field operators:

$$e^{i\hat{H}t_2}\ \hat{\phi}(\vec{x}_2)\ e^{-i\hat{H}t_2} = \hat{\phi}(x_2)$$

$$e^{i\hat{H}t_1}\ \hat{\phi}(\vec{t}_1)\ e^{-i\hat{H}t_1} = \hat{\phi}(x_1)$$

So,

$$= \langle \phi_{\text{final}} |\, e^{-i\hat{H}T}\ \hat{\phi}(x_2)\,\hat{\phi}(x_1)\ e^{-i\hat{H}T}\, | \phi_{\text{init}} \rangle .$$

If $t_1 > t_2$, then the operators would come out the other way. So putting both cases together,

$$\int \mathcal{D}\phi\ \phi(x_1)\,\phi(x_2)\,e^{iS} = \langle \phi_{\text{final}} |\, e^{-i\hat{H}T}\ T\Big(\hat{\phi}(x_2)\,\hat{\phi}(x_1)\Big)\ e^{-i\hat{H}T}\, | \phi_{\text{init}} \rangle$$

Now take limit of both sides as $T \to +\infty(1-i\epsilon)$

adds a little friction

$\Rightarrow$ makes only ground state survive:

$$e^{-i\hat{H}T} | \phi_{\text{init}} \rangle = \sum_n e^{-iE_nT} |n\rangle\langle n | \phi_{\text{init}} \rangle$$

$$\xrightarrow[\infty(1-i\epsilon)]{} e^{-iE_0\infty(1-i\epsilon)} |\Omega\rangle \langle \Omega | \phi_{\text{init}} \rangle$$

Hence,

cancels when divided by $Z = \int \mathcal{D}\phi\, e^{iS}$

$$\lim_{T\to\infty-i\epsilon} \int \mathcal{D}\phi\ \phi(x_1)\,\phi(x_2)\,e^{iS} = e^{-2iE_0\infty(1-i\epsilon)} \langle \Omega | \phi_{\text{init}} \rangle \langle \phi_{\text{final}} | \Omega \rangle$$

$$\times \langle \Omega |\, T\big(\phi(x_1)\,\phi(x_2)\big)\, | \Omega \rangle$$