Perturbation Theory from Functional Integral Formalism

Recall: $$Z[j] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \, G^{[n]}(x_1, \ldots x_n)\, ij(x_1) \ldots ij(x_n)$$

n-point (not necessarily connected) Greens function obtained by taking n derivatives w.r.t. source, at setting $j=0$

$$\frac{1}{Z[0]} \frac{\delta^n}{\delta(ij)^n} Z[j] \Big|_{j=0} = \langle 0| T(\hat{\phi}(x_1) \ldots \hat{\phi}(x_n)) |0\rangle$$

Perturbation theory:

Path integral representation:

$$Z[j] = \int \mathcal{D}\phi \; e^{i\int d^4x (\mathcal{L}[\phi] + j\phi)}$$

Split $\mathcal{L}$ into free part $\mathcal{L}_0$ & interaction part $\mathcal{L}_{int}$:

$$= \int \mathcal{D}\phi \; e^{i\int d^4x (\mathcal{L}_0[\phi] + j\phi)} \, e^{i\int d^4x \, \mathcal{L}_{int}[\phi]}$$

Note taking derivative with respect to $ij(x)$ brings down $\phi(x)$ — so, pull interacting factor to front, and replace $\phi(x)$ with $\frac{\delta}{\delta ij(x)}$:

$$Z[j] = \int \mathcal{D}\phi \; e^{i\int d^4z \, \mathcal{L}_{int}\left[\frac{\delta}{\delta(ij(z))}\right]} \, e^{i\int d^4x (\mathcal{L}_0[\phi] + j\phi)}$$

$$= e^{i\int d^4z \, \mathcal{L}_{int}\left[\frac{\delta}{\delta(ij(z))}\right]} \underbrace{\int \mathcal{D}\phi \; e^{i\int d^4x (\mathcal{L}_0[\phi] + j\phi)}}_{\text{Can perform integration explicitly.}}$$

Example: ϕ^4 theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

then,

$$Z[j] = e^{i\int d^4z \frac{-\lambda}{4!}\left(\frac{\delta}{\delta ij(z)}\right)^4} \int \mathcal{D}\phi \; e^{i\int d^4x \left[\frac{1}{2}\phi(-\partial^2 - m^2)\phi + j\phi\right]}$$

$$= \mathcal{N} \frac{1}{\sqrt{\det(-\partial^2 - m^2)}} \, e^{i\int d^4z \, -\frac{\lambda}{4!}\left(\frac{\delta}{\delta ij(z)}\right)^4} \, e^{-\frac{i}{2}\int d^4x\, d^4y \; j(x) \frac{1}{-\partial^2 - m^2 + i\epsilon} j(y)}$$

Denote the inverse of $-\partial^2 - m^2$ as $-iG_F(x-y)$

$$(-\partial^2 - m^2 + i\epsilon)\left[-iG_F(x-y)\right] = \delta^{(4)}(x-y)$$

Then

$$Z[j] = \frac{1}{\sqrt{\det(-\partial^2 - m^2)}}\, e^{i\int d^4z\, \frac{-\lambda}{4!} \frac{\delta^4}{\delta(ij_z)^4}}\, e^{\frac{-i}{2} j_x (-iG_{xy}) j_y}$$

$$= \frac{1}{\sqrt{\det(-\partial^2 - m^2)}}\, e^{i\int d^4z\, \frac{-\lambda}{4!} \frac{\delta^4}{\delta(ij_z)^4}} \left(e^{-\frac{1}{2} j_x G_{xy} j_y} \right)$$

Usage:

① Expand interaction exponential to desired order of perturbation theory:

$$e^{i\int d^4z\, \frac{-\lambda}{4} \frac{\delta^4}{\delta(ij_z)^4}} = 1 + i\int d^4z\, \frac{-\lambda}{4} \frac{\delta^4}{\delta(ij_z)^4} + \dots,$$

and apply derivatives to exponential at right.

② Differentiate $Z[j]$ with respect to $ij(x)$ n times, and set $j=0$ to access n-point correlation function.

Normalizing factor $\frac{1}{\sqrt{\det(-\partial^2 - m^2)}}$ along with vacuum graphs are removed in the computation of scattering amplitudes via LSZ prescription:

$$G_{conn} = \frac{\langle 0|T\, \phi \dots e^{i\int d^4x\, \mathcal{L}_{int}}|0\rangle_{conn}}{\langle 0|e^{i\int d^4x\, \mathcal{L}_{int}}|0\rangle}$$