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Quantum Equations of Motion (Schwinger–Dyson equations)

$$\langle g.s \vert T(\phi(x_1)\,\phi(x_2)\,\phi(x_3)) \vert g.s\rangle = \frac{1}{Z}\int \mathcal{D}\phi\;\phi(x_1)\phi(x_2)\phi(x_3)\;e^{i\int d^4x\,\mathcal{L}[\phi]}$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$$

In classical mechanics, we stationarize the action, $S[\phi] = \int d^4x\,\mathcal{L}[\phi]$, by looking at an arbitrary infinitessimal variation:

$$\phi(x) \to \phi'(x) = \phi(x) + \epsilon(x) \qquad (*)$$

In the language of Functional integration, $\phi(x) \to \phi'(x)$ is viewed as a change of variables. → Hence, the value of the functional integral is unchanged:

$$\int \mathcal{D}\phi\;\phi(x_1)\phi(x_2)\phi(x_3)\;e^{i\int d^4x\,\mathcal{L}[\phi]} = \underbrace{\int \mathcal{D}\phi'}_{=\int\mathcal{D}\phi \ (\phi\to\phi' \text{ just a shift})}\;\phi'(x_1)\,\phi'(x_2)\,\phi'(x_3)\;e^{i\int d^4x\,\mathcal{L}[\phi']} \qquad (**)$$

Under $(*)$, the Lagrangian changes:

$$\mathcal{L}[\phi'] = \frac{1}{2}\partial_\mu(\phi+\epsilon)\,\partial^\mu(\phi+\epsilon) - \frac{1}{2}m^2(\phi+\epsilon)^2$$

keep $\mathcal{O}(\epsilon)$:

$$= \underbrace{\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2}_{\mathcal{L}[\phi]} + \partial_\mu\phi\,\partial^\mu\epsilon - m^2\phi\epsilon + \mathcal{O}(\epsilon^2)$$

So, RHS of $(**)$ becomes:

$$\int\mathcal{D}\phi\;(\phi(x_1)+\epsilon(x_1))\cdots(\phi(x_3)+\epsilon(x_3))\;e^{i\int d^4x\left(\underbrace{\mathcal{L}[\phi]}_{\text{Factor out (push to back)}} + \underbrace{\partial_\mu\phi\,\partial^\mu\epsilon - m^2\phi\epsilon + \mathcal{O}(\epsilon^2)}_{\text{expand this exp to }\mathcal{O}(\epsilon)}\right)}$$

$$= \int\mathcal{D}\phi\;(\phi(x_1)+\epsilon(x_1))\ldots(\phi(x_3)+\epsilon(x_3))\;e^{i\int d^4x\,[\partial_\mu\phi\,\partial^\mu\epsilon - m^2\phi\epsilon]}\;e^{i\int d^4x\,(\mathcal{L}[\phi])}$$

integrate by parts

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$$= \int \mathcal{D}\phi \, \big(\phi(x_1)+\epsilon(x_1)\big)\cdots\big(\phi(x_3)+\epsilon(x_3)\big)\left[1 + i\int d^4x \;\; \epsilon(-\partial^2 - m^2)\phi\right] \;\; e^{i\int d^4x\, \mathcal{L}[\phi]}$$

$$= \int \mathcal{D}\phi \Big[\underbrace{\phi(x_1)\phi(x_2)\phi(x_3)}_{\text{cancels w/ LHS of } (**)} + \phi(x_1)\phi(x_2)\epsilon(x_3) + \phi(x_1)\epsilon(x_2)\phi(x_3) + \epsilon(x_1)\phi(x_2)\phi(x_3)$$
$$+ i\,\phi(x_1)\phi(x_2)\phi(x_3)\int d^4x \;\; \epsilon(-\partial^2-m^2)\phi \Big] \, e^{i\int d^4x\, \mathcal{L}[\phi]}$$

So $(**)$ to $\mathcal{O}(\epsilon)$ becomes:

$$0 = \int \mathcal{D}\phi \Big[\phi(x_1)\phi(x_2)\epsilon(x_3) + \phi(x_1)\epsilon(x_2)\phi(x_3) + \epsilon(x_1)\phi(x_2)\phi(x_3)$$
$$+ i\,\phi(x_1)\phi(x_2)\phi(x_3)\int d^4x \;\; \epsilon(-\partial^2-m^2)\phi \Big] \, e^{i\int d^4x\, \mathcal{L}[\phi]}$$

Now, write $\epsilon(x_3) = \int d^4x \;\; \delta^{(4)}(x-x_3)\,\epsilon(x)$ to combine first three terms under integral.

$$0 = \int \mathcal{D}\phi \int d^4x \Big[\phi(x_1)\phi(x_2)\delta^{(4)}(x-x_3) + \phi(x_1)\,\delta^{(4)}(x-x_2)\phi(x_3) + \delta^{(4)}(x-x_1)\phi(x_2)\phi(x_3)$$
$$+ i\,\phi(x_1)\phi(x_2)\phi(x_3)\,(-\partial^2-m^2)\phi(x)\Big]\epsilon(x) \;\; e^{i\int d^4x\, \mathcal{L}[\phi]}$$

Since this must be true for any variation $\epsilon(x)$,

$$\int \mathcal{D}\phi \Big[\phi_1\phi_2\,\delta^{(4)}(x-x_3) + \phi_1\,\delta^{(4)}(x-x_2)\phi_3 + \delta^{(4)}(x-x_1)\,\phi_2\phi_3$$
$$+ i\phi_1\phi_2\phi_3\,(-\partial^2-m^2)\phi(x)\Big] \, e^{i\int d^4x\, \mathcal{L}[\phi]}$$

$$\Rightarrow \quad i\,(-\partial^2-m^2)\,\langle g.s.|\,T(\phi_1\phi_2\phi_3\phi_x)\,|g.s.\rangle$$
$$= -\sum_{i=1}^{3} \langle g.s|\,T(\phi_1\cdots\delta(x-x_i)\cdots\phi_3)\,|g.s.\rangle$$

Hence quantum TCFs obey K.G. equations of motion, up to contact terms.

If the field theory is not free, then

$$e^{i\int d^4x\, \mathcal{L}[\phi]} \longrightarrow e^{i\int d^4x'\left(\mathcal{L}[\phi] + \underbrace{\epsilon_x \frac{\delta\mathcal{L}}{\delta\phi_x}}_{\text{space-time integration over repeated indices.}} + \mathcal{O}(\epsilon^2)\right)}$$

$$= \underbrace{e^{i\int d^4x' \int d^4x\, \epsilon(x) \frac{\delta\mathcal{L}[\phi(x')]}{\delta\phi(x)} + \mathcal{O}(\epsilon^2)}}_{\text{expand to } \mathcal{O}(\epsilon):} \underbrace{e^{i\int d^4x' \mathcal{L}[\phi(x')]}}_{\text{Pull to back.}}$$

$$= \left(1 + \int d^4x\; i\,\epsilon(x) \underbrace{\frac{\delta}{\delta\phi(x)} \int d^4x'\, \mathcal{L}[\phi(x')]}_{\text{In free-field case this was } i(-\partial^2 - m^2)}\right) e^{i\int d^4x' \mathcal{L}[\phi(x')]}$$

All other manipulations stay the same, so in the end, we get

$$\boxed{\begin{aligned}&\langle g.s \,|\, T\Big(\Big(i \frac{\delta}{\delta\phi(x)} \underbrace{\int d^4x' \mathcal{L}[\phi(x')]}_{S[\phi]}\Big) \hat{\phi}_1 \cdots \hat{\phi}_n\Big) \,|\, g.s.\rangle \\ &\qquad = -\sum_{i=1}^{n} \langle g.s. \,|\, T\big(\phi_1 \cdots \delta^{(4)}(x-x_i) \cdots \phi_n\big) \,|\, g.s.\rangle\end{aligned}}$$

Schwinger-Dyson equations

$$\langle g.s. |\, T\, \widehat{EOM}(x)\, \underbrace{\hat{\phi}_1 \cdots \hat{\phi}_n \,|\, g.s.\rangle}_{\text{n-point correlation function}} = -\sum_{i=1}^{n} \langle g.s. |\, T\, \underbrace{\hat{\phi}_1 \cdots \delta^{(4)}(x-x_i) \cdots \hat{\phi}_n \,|\, g.s.\rangle}_{\text{(n-1)-point correlation function}}$$

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Special case of

With one power of field:

$$\langle 0|\hat{\phi}(x_1)|0\rangle = \frac{1}{Z}\int \mathcal{D}\phi\ \phi(x_1)\, e^{i\int d^4x\ \mathcal{L}[\phi]}$$

where $\mathcal{L} = \frac{1}{2}\phi(-\partial^2 - m^2)\phi$

shift $\phi \to \phi + \epsilon$ as a change of integration variables.

$$= \frac{1}{Z}\int \mathcal{D}\phi\,(\phi(x_1)+\epsilon(x_1))\, e^{i\int d^4x\left(\mathcal{L}[\phi] + \epsilon(-\partial^2-m^2)\phi + \mathcal{O}(\epsilon^2)\right)}$$

Expand integrand to $\mathcal{O}(\epsilon)$

$$= \frac{1}{Z}\int \mathcal{D}\phi\left[\cancel{\phi(x_1)} + \epsilon(x_1)\, e^{i\int d^4x\,\mathcal{L}[\phi]} + \phi(x_1)\, i\int d^4x\ \epsilon(x)(-\partial^2-m^2)\phi(x)\, e^{i\int d^4x\,\mathcal{L}} + \ldots\right]$$

cancels against that term.

insert $\int d^4x\ \delta(x-x_1)$

$$0 = \frac{1}{Z}\int \mathcal{D}\phi \int d^4x\ \underbrace{\left[\delta(x-x_1) + i\,\phi(x_1)\,(-\partial^2-m^2)\,\phi(x)\right]}_{\text{should vanish.}}\,\epsilon(x)\, e^{i\int d^4x\,\mathcal{L}}$$

$$\langle 0|T\left(\delta^{(4)}(x-x_1) + i\,\hat{\phi}(x_1)(-\partial^2-m^2)\hat{\phi}(x)\right)|0\rangle = 0$$

$$i(-\partial^2-m^2)\underbrace{\langle 0|T\left(\hat{\phi}(x)\,\hat{\phi}(x_1)\right)|0\rangle}_{G_F(x-x_1)} = -\delta^{(4)}(x-x_1)$$

write $D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\,\tilde{D}_F(p)$

$$i(p^2-m^2)\ \tilde{G}_F(p) = -1$$

$$\tilde{G}_F(p) = \frac{i}{p^2-m^2+i\varepsilon}$$

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Quantum Equation of Motion – Fermionic Field

$$\langle g.s.|\, \bar\psi_a(y)\,\psi_b(z)\,|g.s.\rangle = \frac{1}{Z}\int \mathcal{D}\psi\,\mathcal{D}\bar\psi\ \bar\psi_a(y)\, e^{i\int d^4x\,\mathcal{L}[\psi,\bar\psi]}\,\psi_b(z) \qquad (*)$$

$$\mathcal{L} = \bar\psi(i\not\partial - m)\psi$$

Perform shift: $\psi(x) \to \psi'(x) = \psi(x) + \epsilon(x)$ ← 4-component
$\bar\psi(x) \to \bar\psi'(x) = \bar\psi(x) + \bar\epsilon(x)$ ← Grassman-valued

as a change of integration variables.

$$= \frac{1}{Z}\int \mathcal{D}\psi\,\mathcal{D}\bar\psi\ \bar\psi'_a(y)\, e^{i\int d^4x\,\mathcal{L}[\psi',\bar\psi']}\,\psi'_b(z), \qquad (**)$$

where
$$\begin{aligned}\mathcal{L}[\psi',\bar\psi'] &= \bar\psi'(i\not\partial - m)\psi' \\ &= (\bar\psi + \bar\epsilon)(i\not\partial - m)(\psi + \epsilon) \\ &= \underbrace{\bar\psi_a(i\not\partial - m)_{ab}\psi_b}_{\mathcal{L}[\psi,\bar\psi]} + \bar\epsilon_a(i\not\partial - m)_{ab}\psi + \bar\psi_a(i\underset{\uparrow\ \text{Integrate by parts.}}{\not\partial} - m)_{ab}\epsilon_b + O(\epsilon^2)\end{aligned}$$

So path integral $(**)$ becomes:

$$\int \mathcal{D}\psi\,\mathcal{D}\bar\psi\ (\bar\psi(y) + \bar\epsilon(y))_a\, e^{i\int d^4x\left(\mathcal{L}[\psi,\bar\psi] + \underbrace{\bar\epsilon(i\not\partial - m)\psi + \bar\psi(-i\overleftarrow{\not\partial} - m)\epsilon}_{\text{expand to } O(\epsilon)}\right)}\,(\psi(z) + \epsilon(z))_b$$

$$= \int \mathcal{D}\psi\,\mathcal{D}\bar\psi\ (\bar\psi(y) + \bar\epsilon(y))_a\left[1 + i\int d^4x\left(\bar\epsilon(x)(i\not\partial - m)\psi(x) + \bar\psi(x)(-i\overleftarrow{\not\partial} - m)\epsilon(x)\right)\right]$$
$$\times\ e^{i\int d^4x\,\mathcal{L}[\psi,\bar\psi]}\,(\psi(z) + \epsilon(z))_b$$

$$= \int \mathcal{D}\psi\,\mathcal{D}\bar\psi\Big[\bar\psi_a(y)\psi_b(z) + \bar\epsilon_a(y)\psi_b(z) + \bar\psi_a(y)\epsilon_b(z)$$
$$+\, i\,\bar\psi_a(y)\int d^4x\left(\bar\epsilon_c(x)(i\not\partial - m)_{cd}\psi_d(x) + \bar\psi_c(x)(-i\overleftarrow{\not\partial} - m)_{cd}\epsilon_d(x)\right)\psi_b(z)\Big]\, e^{i\int d^4x\,\mathcal{L}[\psi,\bar\psi]}$$

Cancels with $(*)$

Now write $\begin{cases}\bar\epsilon_a(y) = \int d^4x\ \delta^{(4)}(x-y)\,\delta_{ac}\,\bar\epsilon_c(x) \\ \epsilon_b(z) = \int d^4x\ \delta^{(4)}(x-z)\,\delta_{bd}\,\epsilon_d(x)\end{cases}$ to combine terms.

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passing ϵ through $\bar\psi$

$$0 = \int \mathcal{D}\psi \mathcal{D}\bar\psi \int d^4x \left\{ \bar\epsilon_c(x) \left[\delta^{(4)}(x-y)\, \delta_{ac}\, \psi_b(z) - i \bar\psi_a(y)\, (i\not{\partial}-m)_{cd}\, \psi_d(x)\, \psi_b(z) \right] \right.$$

$$\left. + \left[\bar\psi_a(y)\, \delta^{(4)}(x-z)\, \delta_{bd} - i \bar\psi_a(y)\, \bar\psi_c(x)\, (-i\overleftarrow{\not{\partial}}-m)_{cd}\, \psi_b(z) \right] \epsilon_d(x) \right\} e^{i\int d^4x \mathcal{L}}$$

Since equation must hold for any $\epsilon(x)$ and $\bar\epsilon(x)$, we have:

$$0 = \int \mathcal{D}\psi \mathcal{D}\bar\psi \left[\delta^{(4)}(x-y)\, \delta_{ac}\, \psi_b(z) - i \bar\psi_a(y)\, (i\not{\partial}-m)_{cd}\, \psi_d(x)\, \psi_b(z) \right] e^{i\int d^4x \mathcal{L}[\bar\psi,\psi]}$$

$$0 = \int \mathcal{D}\psi \mathcal{D}\bar\psi \left[\bar\psi_a(y)\, \delta^{(4)}(x-z)\, \delta_{bd} - i \bar\psi_a(y)\, \bar\psi_c(x)\, (-i\overleftarrow{\not{\partial}}-m)_{cd}\, \psi_b(z) \right] e^{i\int d^4x \mathcal{L}[\bar\psi,\psi]}$$

$$\Rightarrow i\, (i\not{\partial}-m)_{cd}\, \langle g.s. | T\left(-\psi_d(x)\, \bar\psi_a(y)\, \psi_b(z)\right) | g.s.\rangle = \delta^{(4)}(x-y)\, \delta_{ac}\, \langle g.s. | \psi_b(z) | g.s.\rangle$$

$$\Rightarrow i \langle g.s. | T\left(-\bar\psi_a(y)\, \psi_b(z)\, \bar\psi_c(x)\right) | g.s.\rangle\, (-i\overleftarrow{\not{\partial}}-m)_{cd} = \delta^{(4)}(x-z)\, \delta_{bd}\, \langle g.s. | \bar\psi_a(y) | g.s.\rangle$$

<u>OR</u> (after omitting fields $\psi_b(z)$ in first eqn
$\bar\psi_a(y)$ in second eqn)

$$\boxed{\begin{aligned} (i\not{\partial}-m)_{cd}\, \langle g.s. | T\left(\psi_d(x)\, \bar\psi_a(y)\right) | g.s.\rangle &= i\, \delta^{(4)}(x-y)\, \delta_{ac} \\ \langle g.s. | T\left(\psi_b(z)\, \bar\psi_c(x)\right) | g.s.\rangle\, (-i\overleftarrow{\not{\partial}}-m)_{cd} &= i\, \delta^{(4)}(x-z)\, \delta_{bd} \end{aligned}}$$