Bessel Function Representation for Thermal Functions

Thermal Function:

$$J_{B/F}(z^2) = \int_0^\infty dx\, x^2 \ln\left(1 \mp e^{-\sqrt{x^2+z^2}}\right)$$

ch var: $x \to \sqrt{z^2}\, x'$

$$= (z^2)^{3/2} \int_0^\infty dx'\, (x')^2 \ln\left(1 \mp e^{-\sqrt{z^2}\sqrt{(x')^2+1}}\right)$$

ch var: $\sqrt{(x')^2+1} \longrightarrow x \qquad x'dx' = x\,dx \qquad y^2 - 1 = x'$

$$= (z^2)^{3/2} \int_0^\infty \sqrt{x^2-1}\; x\,dx\, \ln\left(1 \mp e^{-\sqrt{z^2}\,x}\right)$$

Perform Taylor expansion: $\ln(1 \mp \epsilon) = -\sum_{n=1}^{\infty} \frac{(\pm 1)^n}{n} \epsilon^n, \quad -1 < \epsilon < +1.$

Since $e^{-\sqrt{z^2}\,x} < 1$,

$$J_{B/F}(z^2) = (z^2)^{3/2} \int_0^\infty \sqrt{x^2-1}\; x\,dx \left(-\sum_{n=1}^{\infty} \frac{(\pm 1)^n}{n} e^{-\sqrt{z^2}\,xn}\right)$$

$$= -\sum_{n=1}^{\infty} (z^2)^{3/2} \frac{(\pm 1)^n}{n} \int_0^\infty \sqrt{x^2-1}\; x\,dx\, e^{-\sqrt{z^2}\,xn}$$

$$= -\sum_{n=1}^{\infty} (z^2) \frac{(\pm 1)^n}{n^2} \left(zn \int_0^\infty \underbrace{\sqrt{x^2-1}\; x}_{\frac{1}{3x}\frac{d}{dx}(x^2-1)^{3/2}}\,dx\, e^{-\sqrt{z^2}\,xn} \right)$$

$$= \quad '' \quad \frac{zn}{3} \int_0^\infty \frac{d}{dx}(x^2-1)^{3/2}\,dx\, e^{-\sqrt{z^2}\,xn}$$

$$= \quad '' \quad -\frac{zn}{3} \int_0^\infty (x^2-1)^{3/2} \frac{d}{dx} e^{-\sqrt{z^2}\,xn}\,dx$$

$$= \quad '' \quad +\frac{z^2n^2}{3} \int_0^\infty (x^2-1)^{3/2} e^{-\sqrt{z^2}\,xn}\,dx = K_2(\sqrt{z^2}\,n)$$

Modified Bessel function of the 2^{nd} kind.
[See Mathworld (eqn 7) set $n=2$]

$$\boxed{J_{B/F}(z^2) = -\sum_{n=1}^{\infty} \frac{(\pm 1)^n z^2}{n^2} K_2(\sqrt{z^2}\,n)}$$

Derivative:

$$J'_{B/F}(z^2) = -\sum_{n=1}^{\infty}\left[\frac{(\pm1)^n}{n^2}K_2(\sqrt{z^2}\,n) + \frac{(\pm1)^n\,\cancel{(z^2)^{1/2}}}{n^2}K_2'(\sqrt{z^2}\,n)\times\frac{1}{2}\frac{1}{\cancel{\sqrt{z^2}}}n\right]$$

$$= -\sum_{n=1}^{\infty}\frac{(\pm1)^n}{n^2}\left[K_2(\sqrt{z^2}\,n) + n\frac{\sqrt{z^2}}{2}\left(-\frac{K_1(\sqrt{z^2}\,n)}{2} - \frac{K_3(\sqrt{z^2}\,n)}{2}\right)\right]$$

$$= \sum_{n=1}^{\infty}\frac{(\pm1)^n\sqrt{z^2}}{4n}\left[\frac{-4}{n\sqrt{z^2}}K_2(\sqrt{z^2}\,n) + K_1(\sqrt{z^2}\,n) + K_3(\sqrt{z^2}\,n)\right]$$