Path Integral Formulation of Thermal Field Theory (imaginary time)

In statistical mechanics, thermodynamic quantities are derived from a partition function:

$$Z(\beta) = \mathrm{Tr}\left[e^{-\beta\hat{H}}\right], \qquad \beta = \frac{1}{k_B T} > 0.$$

$$= \sum_{\text{States}, n} \langle n| e^{-\beta\hat{H}} |n\rangle$$

Let us choose to sum over states that are of definite field configuration.

$$= \int \mathcal{D}\phi_0 \underbrace{\langle\phi_0| e^{-\beta\hat{H}} |\phi_0\rangle}_{\substack{\text{If } \beta = +i(t_f - t_i)\text{, we} \\ \text{know how to calculate this:}}} \qquad (*)$$

$$\langle\phi_0| e^{-i\hat{H}(t_f - t_i)} |\phi_0\rangle = \int_{\phi(t_i)=\phi_0}^{\phi(t_f)=\phi_0} \mathcal{D}\phi \; e^{i\int_{t_i}^{t_f} dt \int d^3x\, \mathcal{L}[\phi, \partial_\mu\phi, \ldots]}$$

However, what we need is $e^{-\beta\hat{H}}$.

- Can get this by Euclideanizing the theory — later

See I&Z pg. 299 "Eucl. Green's Fn."

Essentially, by putting $t = -i\beta$, we can take $\mathcal{L}[\phi, \partial_\mu\phi, \ldots] \longrightarrow \mathcal{L}_E[\phi, \partial_\mu\phi, \ldots]$.
(Wick rotation)

"Euclideanized Lagrangian"

Result: $\langle\phi_0| e^{-\beta\hat{H}} |\phi_0\rangle = \int_{\phi(0)=\phi_0}^{\phi(\beta)=\phi_0} \mathcal{D}\phi \; e^{-i\int_0^\beta d\tau \int d^3x\, \mathcal{L}_E[\phi, \partial_\mu\phi, \ldots]}$

So, finally (plugging this into $(*)$)

$$Z(\beta) = \underbrace{\int \mathcal{D}\phi_0 \int_{\phi(0)=\phi_0}^{\phi(\beta)=\phi_0} \mathcal{D}\phi}\; e^{-\overbrace{\int_0^\beta d\tau \int d^3x\, \mathcal{L}_E[\phi, \partial_\mu\phi, \ldots]}^{\text{Abbreviate as } S_E[\phi] \text{ (Euclideanized Action)}}}$$

$$= \int_{\text{PBC}} \mathcal{D}\phi \; e^{-S[\phi]}$$

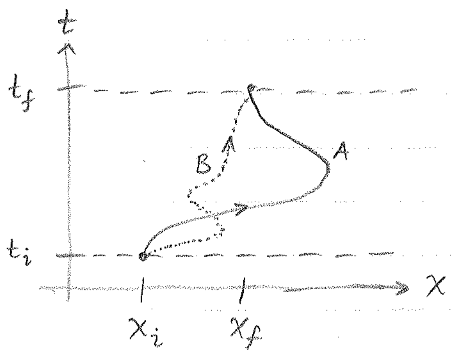
PBC ≡ "Periodic Boundary condition"

What is $\displaystyle\int_{PBC} \mathcal{D}\phi = \int \mathcal{D}\phi_0 \int_{\phi(t_i)=\phi_0}^{\phi(t_f)=\phi_0} \mathcal{D}\phi$?

In elementary quantum mechanics (0+1 dimensional QFT), we encounter functional integrals of the form:

$$\underbrace{\int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}x}\; e^{-iS[x]} \equiv \langle x(t_f) | x(t_i) \rangle$$

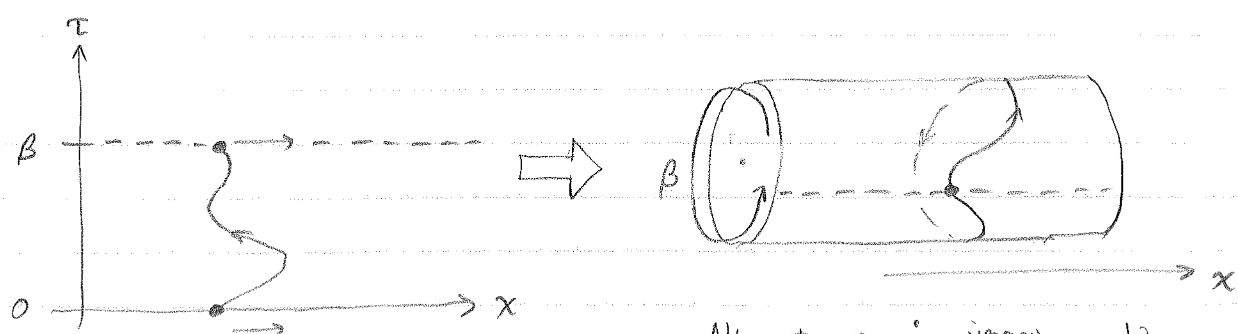
This instructs us to sum up all contributions from paths $x(t)$ that start at $x(t_i)=x_i$ and end at $x(t_f)=x_f$.



Two possible paths, A & B, that contribute to the path integral.

Hence, the inner functional integral, $\int_{\phi(\tau_i)=\phi_0}^{\phi(t_f)=\phi_0} \mathcal{D}\phi$, does just that, but with the initial and end-points at the same location (field configuration).

The outer integral instructs us to repeat the same task with all possible initial (and final) configurations, and sum over all contributions.



push end points along x-axis (and sum up all contributions)

Alternate view: imaginary time direction compactified (dimension size inversely prop. to temperature). Sum over all loops (Periodic boundary condition).

We would like to calculate thermodynamic (ensemble) averages of observables, eg. $\hat{A}$. Since all observable operators may be built out of field operators, add a source (as in zero temperature field theory).

$$\mathcal{L}_E[\phi, \partial_\mu\phi] \longrightarrow \mathcal{L}_E[\phi, \partial_\mu\phi] + j\phi .$$

Then, the partition function becomes

$$Z_\beta[j] = \int_{PBC} \mathcal{D}\phi \; e^{-\int_0^\beta d\tau \int d^3x \left(\mathcal{L}_E[\phi, \partial_\mu\phi, \dots] + j\phi\right)}$$

The ensemble averages can be found by taking functional derivatives:

for example:

$$\langle \phi(\vec{y})\, \phi(\vec{x}) \rangle_\beta = \left(\frac{-\delta}{\delta j(\vec{y})}\right)\left(\frac{-\delta}{\delta j(\vec{x})}\right) Z_\beta[j] .$$

n.b: If the $j(\vec{x})$ in functional derivatives have non-zero imaginary time argument, the resulting correlation will be (inverse) temperature ordered.

— such correlation functions are not physical.

For correlation functions in real time, have to perform the difficult analytic continuation, or use the "Real Time" formalism.