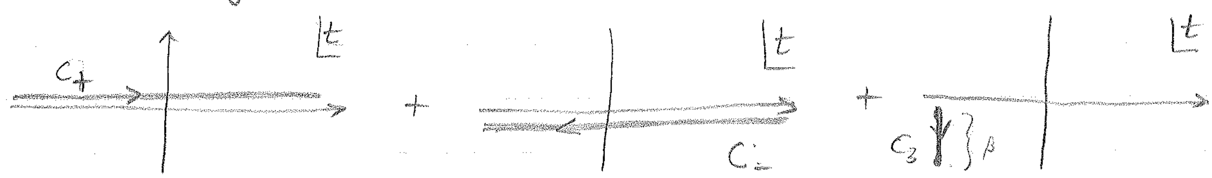
Introduce a generating functional of the form:

$$Z[J_c] = \int_{PBC} \mathcal{D}\phi \; e^{i\int_c dt \int d^3x \,(\mathcal{L} + J_c \phi)}$$

where we take $\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$.

The dt integration contour is



Together, defines a close contour in complex time plane

negligible – check this.

That is $\int_c dt = \int_{-\infty + i\epsilon}^{+\infty + i\epsilon} dt_+ - \int_{-\infty - i\epsilon}^{+\infty - i\epsilon} dt_-$

infinitessimally displaced above real axis

inf. displaced below real axis

Obtain Greens functions by taking functional derivatives.
For example, two-point Green's Function:

$$iG_c(t-t') = \langle T_c(\phi(t), \phi(t')) \rangle$$

$$= \theta_c(t-t')\langle \phi(t)\phi(t')\rangle + \theta_c(t'-t)\langle \phi(t')\phi(t)\rangle$$

$$= \left(-i\frac{\delta}{\delta J_c(t)}\right)\left(-i\frac{\delta}{\delta J_c(t)}\right) Z[J_c]\Big|_{J_c=0}$$

By $J_c(t)$, we mean the source evaluated at time t. Since the two contours C_+ & C_- are infinitessimally displaced from the real time axis, we must specify this:

$$J_c(t+i\epsilon) \neq J_c(t-i\epsilon)$$

on C_+ on C_-

The T_c means "path ordered",

$$\theta_c(t-t') = \begin{cases} \theta(t-t') & t,t' \text{ on } C_+ \\ \theta(t'-t) & t,t' \text{ on } C_- \\ 0 & t \text{ on } C_+,\ t' \text{ on } C_- \\ 1 & t' \text{ on } C_-,\ t \text{ on } C_+ \end{cases}$$

times on C_- come after times of C_+. Define Dirac delta by:

$$\delta_c(t-t') = \frac{d\theta_c(t-t')}{dt} = \begin{cases} \delta(t-t') & t,t' \text{ on } C_+ \\ -\delta(t-t') & t,t' \text{ on } C_- \\ 0 & \end{cases}$$

If the field theory is non-interacting, the generating functional can be readily integrated

$$Z[J_c] = \underbrace{\left(\frac{1}{\mathrm{Det}[\partial^2 - m^2]}\right)^{1/2}}_{\text{Unimportant normalization constant}} \exp\left[\frac{-i}{2}\int_c dt\,dt' \int d^3x\, d^3y\; J_c(\vec{x},t)\, G_c(\vec{x}-\vec{y},t-t')\, J_c(\vec{y},t')\right]$$

The resulting Greens function obeys $(\partial_\mu \partial^\mu + m^2)\, G_c(\vec{x}-\vec{y}, t-t') = -\delta^{(3)}(\vec{x}-\vec{y})\,\delta_c(t-t')$

In 3-momentum space,

$$\left(\frac{\partial^2}{\partial t^2} + \sqrt{\vec{k}^2+m^2}\right) G_c(t-t', \sqrt{\vec{k}^2+m^2}) = -\delta_c(t-t')$$

Solution: (subject to periodic BC) set $\omega_{\vec{k}} = \sqrt{\vec{k}^2+m^2}$

Show this

$$G_c(t-t', \omega_{\vec{k}}) = \frac{n_B(\omega_{\vec{k}})}{2i\omega_{\vec{k}}}\Big[\theta_c(t-t')\left(e^{\beta\omega - i\omega(t-t')} + e^{i\omega(t-t')}\right)$$
$$+ \theta_c(t'-t)\left(e^{-i\omega(t-t')} + e^{\beta\omega + i\omega(t-t')}\right)\Big]$$

$$n_B(\omega_{\vec{k}}) = \frac{1}{e^{\beta\omega_{\vec{k}}} - 1}$$

Properties:

1. As $\beta \to \infty$ (zero temp. limit), $G_c(t-t', \omega) \to G_F(t-t', \omega)$

   Propagator goes to conventional $T=0$, G_F, on C_+.

2. $G_c(t-t') = G_c(t'-t)$

3. Satisfies the KMS

If one of t or t' is on either C_+ or C_-, and the other time, t or t', is on C_3, then the propagator vanishes.

Because if t' is on C_3, $t' = T - is$, where $0 > s > -\beta$.

Then
$$\lim_{T\to-\infty} G_c(t - T + is, \omega)$$
$$= \lim_{T\to-\infty} e^{-i\omega(t-T+is)} + e^{\beta\omega + i\omega(t-T+is)}$$
$$= 0.$$

$\Rightarrow$ Propagator vanishes (doesn't connect points) when one is on C_3, and the other on C_+ or C_-. $\Rightarrow$ This part becomes unimportant.