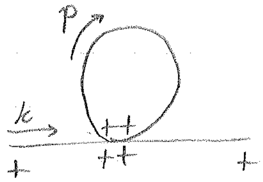
Real-time Self Energy Calculation

$$\mathcal{L}[\phi_+, \phi_-] = \mathcal{L}(\phi_+) - \mathcal{L}(\phi_-),$$

where $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$



$$= -i\Sigma(k^2)$$

$$= \underbrace{4\times 3\times\frac{-i\lambda}{4!}}\int\frac{d^4p}{(2\pi)^4}\Bigg[\underbrace{\frac{i}{p^2-m^2+i\epsilon}}_{\text{Zero-temperature part. UV-div.}} + \underbrace{2\pi\, n_B(|p^0|)\,\delta(p^2-m^2)}_{\text{Finite-temperature part}}\Bigg]$$

$$\begin{pmatrix}\text{Finite-temp}\\ \text{part}\end{pmatrix} = \frac{-i\lambda}{2}\int\frac{dp^0}{(2\pi)}\int\frac{d^3p}{(2\pi)^3}\, 2\pi\, n_B(|p^0|)\,\delta(p^2-m^2)$$

recall, $$\delta(f(x)) = \sum_{\substack{\text{roots of}\\ f(x)=0,\ x_i}}\frac{1}{\left|\frac{\partial f}{\partial x}\right|_{x=x_i}}\times\delta(x-x_i).$$

So, write $$\delta(p^2-m^2) = \frac{1}{|2\omega_{\vec p}|}\,\delta(p^0-\omega_p) + \frac{1}{|-2\omega_{\vec p}|}\,\delta(p^0+\omega_{\vec p})$$

$\omega_{\vec p} \equiv \sqrt{\vec p^{\,2}+m^2}$

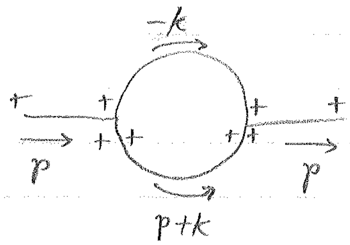
$$\begin{pmatrix}\text{Finite-temp}\\ \text{part}\end{pmatrix} = \frac{-i\lambda}{2}\int\frac{dp^0}{\cancel{(2\pi)}}\int\frac{d^3p}{(2\pi)^3}\,\cancel{2\pi}\, n_B(|p^0|)\left[\frac{1}{2\omega_{\vec p}}\delta(p^0-\omega_p) + \frac{1}{2\omega_{\vec p}}\delta(p^0+\omega_p)\right]$$

$$= \frac{-i\lambda}{2}\int\frac{d^3p}{(2\pi)^3}\left[n_B(|\omega_{\vec p}|)\frac{1}{2\omega_{\vec p}} + n_B(|-\omega_{\vec p}|)\frac{1}{2\omega_{\vec p}}\right]$$

$$= \frac{-i\lambda}{2}\int\frac{d^3p}{(2\pi)^3}\,\frac{n_B(\omega_{\vec p})}{\omega_{\vec p}} = \frac{-i\lambda}{2}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\omega_{\vec p}\left(e^{\beta\omega_{\vec p}}-1\right)},$$

in agreement with imaginary-time formalism.

Real-time calculation of $\Sigma(p^2)$ in ϕ^3 theory.



Propagator: $= \dfrac{i}{k^2-m^2+i\epsilon} + 2\pi\, n_B(|k^0|)\,\delta(k^2-m^2)$

$$-i\Sigma(p^2) = 4\,\frac{1}{2!}\left(\frac{-ig}{2}\right)^2 \int \frac{d^4k}{(2\pi)^4}\left[\frac{i}{k^2-m^2+i\epsilon} + 2\pi\, n_B(|k^0|)\,\delta(k^2-m^2)\right]$$
$$\times\left[\frac{i}{(k+p)^2-m^2+i\epsilon} + 2\pi\, n_B(|k^0+p^0|)\,\delta((k+p)^2-m^2)\right]$$

We want the real part of $-\Sigma(p^2)$.

$$\mathrm{Re}\left[-\Sigma'(p^2)\right] = -i\,\frac{-g^2}{2}\int\frac{d^4k}{(2\pi)^4}\left(\frac{2\pi i\, n_B(|k^0+p^0|)\,\delta((k+p)^2-m^2)}{k^2-m^2+i\epsilon} + \frac{2\pi i\, n_B(|k^0|)\,\delta(k^2-m^2)}{(k+p)^2-m^2+i\epsilon}\right)$$

$$= \frac{-g^2}{2}\int\frac{d^4k}{(2\pi)^4}\,2\pi\left(\frac{n_B(|k^0+p^0|)\,\delta((k+p)^2-m^2)}{k^2-m^2+i\epsilon} + \frac{n_B(|k^0|)\,\delta(k^2-m^2)}{(k+p)^2-m^2+i\epsilon}\right)$$

Write $\delta(k^2-m^2) = \dfrac{1}{2\omega_k}\left(\delta(k^0-\omega_k)+\delta(k^0+\omega_k)\right)$

$$\delta((k+p)^2-m^2) = \frac{1}{2\omega_{k+p}}\left(\delta(k^0+p^0-\omega_{k+p}) + \delta(k^0+p^0+\omega_{k+p})\right)$$

$$= \frac{-g^2}{2}\int\frac{d^3k}{(2\pi)^3}\int\frac{dk^0}{\cancel{2\pi}}\cancel{2\pi}\left(\frac{n_B(|k^0+p^0|)}{(k^0)^2\underbrace{-\vec{k}^2-m^2}_{-\omega_k^2}}\;\frac{1}{2\omega_{k+p}}\Big(\delta(k^0+p^0-\omega_{k+p})+\delta(k^0+p^0+\omega_{k+p})\Big)\right.$$

$$\left.+\frac{n_B(|k^0|)}{(k^0+p^0)^2\underbrace{-(\vec{k}+\vec{p})^2-m^2}_{-\omega_{k+p}^2}}\;\frac{1}{2\omega_k}\Big(\delta(k^0-\omega_k)+\delta(k^0+\omega_k)\Big)\right)$$

Integrate over k^0.

$$= \frac{-g^2}{2}\int\frac{d^3k}{(2\pi)^3}\left[\frac{n_B(\omega_{k+p})}{2\omega_{k+p}}\left(\frac{1}{(\omega_{k+p}+p^0)^2-\omega_k^2}+\frac{1}{(\omega_{k+p}-p^0)^2-\omega_k^2}\right)\right. \quad ②$$

$$\left.+\frac{n_B(\omega_k)}{2\omega_k}\left(\frac{1}{(\omega_k+p^0)^2-\omega_{k+p}^2}+\frac{1}{(\omega_k-p^0)^2-\omega_{k+p}^2}\right)\right] \quad ①$$

$$= \frac{g^2}{2}\int\frac{d^3k}{(2\pi)^3}\left[\frac{n_B(\omega_k)}{2\omega_k}\;\frac{1}{\omega_{k+p}^2-(\omega_k+p^0)^2}+\frac{n_B(\omega_{k+p})}{2\omega_{k+p}}\;\frac{1}{\omega_k^2-(\omega_k+p^0)^2}+(p^0\to -p^0)\right.$$

Now write $\quad n_B(\omega_k) = \dfrac{\coth\left(\frac{\beta\omega_k}{2}\right)-1}{2}$

$$= \frac{g^2}{2}\int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{4}\;\frac{1}{\omega_k}\left(\coth\left(\frac{\beta\omega_k}{2}\right)-1\right)\frac{1}{\omega_{k+p}^2-(\omega_k+p^0)^2}\right.$$

$$\left.+\frac{1}{4}\;\frac{1}{\omega_{k+p}}\left(\coth\left(\frac{\beta\omega_{k+p}}{2}\right)-1\right)\frac{1}{\omega_k^2-(\omega_k+p^0)^2}+(p^0\to -p^0)\right]$$

$$= \frac{g^2}{8}\int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{\omega_k}\coth\left(\frac{\beta\omega_k}{2}\right)\frac{1}{\omega_{k+p}^2-(\omega_k+p^0)^2}\right.$$

$$\left.+\frac{1}{\omega_{k+p}}\coth\left(\frac{\beta\omega_{k+p}}{2}\right)\frac{1}{\omega_k^2-(\omega_k+p^0)^2}+(p^0\to -p^0)\right]+\begin{pmatrix}T=0\\ \text{part}\end{pmatrix}$$

in agreement with Matsubara formalism calculation.

$$[\omega_k^2-(\omega_k-ip^0)^2][\omega_k^2-(\omega_k+ip^0)^2]$$

$$=\omega_k^4-\omega_k^2(\omega_k+ip^0)^2-\omega_k^2(\omega_k-ip^0)^2+(\omega_k-ip^0)^2(\omega_k+ip^0)^2$$

$$(\omega_k^2+\cancel{2ip^0\omega_k}-(p^0)^2)$$

$$=\omega_k^4+2(-\omega_k^4+\omega_k^2(p^0)^2)+(\omega_k^4+2ip^0\omega_k^3-\omega_k^2(p^0)^2$$

Take limits:

$$-\Sigma(p^0,\vec{p}\to 0)=\frac{g^2}{8}\int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{\omega_k}\coth\left(\frac{\beta\omega_k}{2}\right)\frac{1}{\omega_k^2-(\omega_k-ip^0)^2}\right.$$

$$\left.+\frac{1}{\omega_k}\coth\left(\frac{\beta\omega_k}{2}\right)\frac{1}{\omega_k^2-(\omega_k-ip^0)^2}+(p^0\to -p^0)\right]$$

$$=\frac{g^2}{8}\int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{\omega_k}\coth\left(\frac{\beta\omega_k}{2}\right)\underbrace{\left(\frac{1}{\omega_k^2-(\omega_k-ip^0)^2}+\frac{1}{\omega_k^2-(\omega_k+ip^0)^2}\right)}_{\frac{2}{(p^0)^2+4\omega_k^2}}\right]\times 2$$

$$=\frac{g^2}{8}\int\frac{d^3k}{(2\pi)^3}\left[\frac{4}{\omega_k}\coth\left(\frac{\beta\omega_k}{2}\right)\frac{1}{(p^0)^2+4\omega_k^2}\right]$$

Then,

$$-\Sigma(p^0=0,\vec{p}\to 0)=\frac{g^2}{8}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k^3}\overbrace{\coth\left(\frac{\beta\omega_k}{2}\right)}^{(1+2n_B(\omega_k))}.$$

$$-\Sigma(p^0=0,\vec{p})=\frac{g^2}{8}\int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{\omega_k}\coth\left(\frac{\beta\omega_k}{2}\right)\frac{1}{\omega_{k+p}^2-\omega_k^2}+\frac{1}{\omega_{k+p}}\coth\left(\frac{\beta\omega_{k+p}}{2}\right)\frac{1}{\omega_k^2-\omega_{k+p}^2}\right]\times 2$$

$$=\frac{g^2}{8}\int\frac{d^3k}{(2\pi)^3}\left[\underbrace{\frac{1}{\omega_{k+p}^2-\omega_k^2}}_{\frac{1}{0}}\underbrace{\left(\frac{1}{\omega_k}\coth\left(\frac{\beta\omega_k}{2}\right)-\frac{1}{\omega_{k+p}}\coth\left(\frac{\beta\omega_{k+p}}{2}\right)\right)}_{0}\right]$$

$\vec{p}\to 0$ $\qquad = \frac{0}{0}.$

$$=\frac{g^2}{8}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k^3}\left(\coth\left(\frac{\beta\omega_k}{2}\right)+\frac{\beta\omega_k}{2}\operatorname{csch}^2\left(\frac{\beta\omega_k}{2}\right)\right)$$

$1+2n_B(\omega_k)$

$$-\Sigma(\underset{\text{second}}{p^0=0},\ \underset{\text{first}}{\vec{p}=0}) = \frac{g^2}{8}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k^3}\left(1+\frac{2}{e^{\beta\omega_k}-1}\right)$$

$$= \frac{g^2}{8}\int\frac{d^3k}{(2\pi)^3}\frac{1}{(\sqrt{\vec{k}^2+m^2})^3}\frac{2}{e^{\beta\sqrt{\vec{k}^2+m^2}}-1} + \left(\begin{matrix}\text{zero}\\ \text{temp}\end{matrix}\right)$$

$$= \frac{g^2}{8}\frac{4\pi}{(2\pi)^3}\,2\int d|\vec{k}|\,|\vec{k}|^2\frac{1}{(|\vec{k}|^2+m^2)^{3/2}}\frac{1}{e^{\beta\sqrt{\vec{k}^2+m^2}}-1}$$

"IR divergent" in HTL

$$-\Sigma(\underset{\text{first}}{p^0=0},\ \underset{\text{second}}{\vec{p}\to 0}) = \text{above} + \frac{g^2}{8}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k^3}\frac{\beta\omega_k}{2}\underbrace{\operatorname{csch}^2\left(\frac{\beta\omega_k}{2}\right)}_{\text{extraction of T-dep part?}}$$

$$\frac{g^2}{8}\int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{\omega_{k+p}^2-\omega_k^2}\left(\frac{1}{\omega_k}\coth\left(\frac{\beta\omega_k}{2}\right)-\frac{1}{\omega_{k+p}}\coth\left(\frac{\beta\omega_{k+p}}{2}\right)\right)\right]$$

Extract finite temp part:

$$\frac{g^2}{8}\int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{\omega_{k+p}^2-\omega_k^2}\left(\frac{1}{\omega_k}2n_B(\omega_k)-\frac{1}{\omega_{k+p}}2n_B(\omega_{k+p})\right)\right]$$

$$\frac{g^2}{4}\int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{\omega_{k+p}^2-\omega_k^2}\left(\frac{n_B(\omega_k)}{\omega_k}-\frac{n_B(\omega_{k+p})}{\omega_{k+p}}\right)\right] \qquad \begin{matrix}\omega_k=\sqrt{|\vec{k}|+m^2}\\ \omega_{k+p}=\sqrt{|\vec{k}+\vec{p}|+m^2}\end{matrix}$$

take $\vec{p}\to 0$ limit:

$$\longrightarrow \frac{g^2}{4}\int\frac{d^3k}{(2\pi)^3}\frac{1}{n_B^2(\omega_k)\,\omega_k^2}\left[(\omega_k\beta-1)e^{\omega_k\beta}-1\right]$$