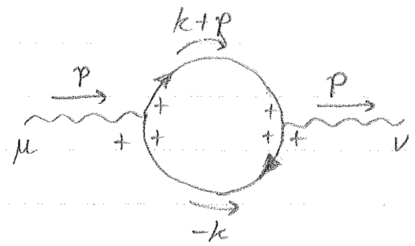
Electron Contribution to Photon Debye Mass



$$i\Pi_{\mu\nu} = -i\mu^{2\epsilon}\int \frac{d^d k}{(2\pi)^d} \,\mathrm{Tr}\left[(-ie\gamma_\mu)(\not k+m)(-ie\gamma_\nu)(\not k+\not p+m)\right]$$

$$\times\left(\frac{1}{k^2-m^2} + 2\pi n_F(|k^0|)\,\delta(k^2-m^2)\right)\left(\frac{1}{(k+p)^2-m^2} + 2\pi n_F(|k^0+p^0|)\,\delta((k+p)^2-m^2)\right)$$

Finite temperature part :

$$= \begin{pmatrix}\text{zero-}\\ \text{temperature}\end{pmatrix} + ie^2\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma_\mu(\not k+m)\gamma_\nu(\not k+\not p+m)\right]$$

$$\times\left(\frac{2\pi n_F(|k^0+p^0|)\,\delta((k+p)^2-m^2)}{k^2-m^2+i\epsilon} + \frac{2\pi n_F(|k^0|)\,\delta(k^2-m^2)}{(k+p)^2-m^2+i\epsilon}\right)$$

Evaluate trace

$$\mathrm{Tr}\left[\gamma_\mu(\not k+m)\gamma_\nu(\not k+\not p+m)\right] = \mathrm{Tr}\left[\gamma_\mu\not k\gamma_\nu(\not k+\not p)\right] + m^2\,\mathrm{Tr}\left[\gamma_\mu\gamma_\nu\right]$$

$$= 4\left(k_\mu(k+p)_\nu - g_{\mu\nu}\,k.(k+p) + (k+p)_\mu k_\nu\right) + \underbrace{m^2\times 4g_{\mu\nu}}$$

In HTL, p^μ & p^ν do not lead to thermal masses.

To obtain HTL, drop this; we want something prop. to T^2.

Get ready for k^0 integration:

$$\frac{1}{k^2-m^2+i\epsilon} = \frac{1}{2\omega_k}\left(\frac{1}{k^0-\omega_k} - \frac{1}{k^0+\omega_k}\right) - i\pi\,\delta((k^0)^2-\omega_k^2)$$

$$\frac{1}{(k+p)^2-m^2+i\epsilon} = \frac{1}{2\omega_{k+p}}\left(\frac{1}{k^0+p^0-\omega_{k+p}} - \frac{1}{k^0+p^0+\omega_{k+p}}\right) - i\pi\,\delta((k^0+p^0)^2-\omega_{k+p}^2)$$

AND

$$\omega_k = \sqrt{\vec{k}^2+m^2}$$

$$\omega_{k+p} = \sqrt{(\vec{k}+\vec{p})^2+m^2}$$

$$\delta((k+p)^2-m^2) = \frac{1}{2\omega_{k+p}}\left(\delta(k^0+p^0-\omega_{k+p}) + \delta(k^0+p^0+\omega_{k+p})\right)$$

$$\delta(k^2-m^2) = \frac{1}{2\omega_k}\left(\delta(k^0-\omega_k) + \delta(k^0+\omega_k)\right)$$

AND

$$\delta\left((k^0)^2-\omega_k^2\right) = \frac{1}{2\omega_k}\left(\delta(k^0-\omega_k)+\delta(k^0+\omega_k)\right)$$

$$\delta\left((k^0+p^0)^2-\omega_{k+p}^2\right) = \frac{1}{2\omega_{k+p}}\left(\delta(k^0+p^0-\omega_{k+p})+\delta(k^0+p^0+\omega_{k+p})\right)$$

For zero mode thermal masses, take $p^0 \to 0$.

$$i\Pi_{\mu\nu}^{HTL} = ie^2\int\frac{d^3k}{(2\pi)^3}\int\frac{dk^0}{2\pi}\, 4(2k_\mu k_\nu - k^2 g_{\mu\nu}) \times 2\pi\, n_F(|k^0|)$$
$$\times\left\{\frac{1}{2\omega_{k+p}}\left(\delta(k^0-\omega_{k+p})+\delta(k^0+\omega_{k+p})\right)\left[\frac{1}{2\omega_k}\left(\frac{1}{k^0-\omega_k}-\frac{1}{k^0+\omega_k}\right) - \frac{i\pi}{2\omega_k}\left(\delta(k^0-\omega_k)+\delta(k^0+\omega_k)\right)\right]\right.$$
$$\left.\frac{1}{2\omega_k}\left(\delta(k^0-\omega_k)+\delta(k^0+\omega_k)\right)\left[\frac{1}{2\omega_{k+p}}\left(\frac{1}{k^0-\omega_{k+p}}-\frac{1}{k^0+\omega_k}\right) - \frac{i\pi}{2\omega_{k+p}}\left(\delta(k^0-\omega_{k+p})+\delta(k^0+\omega_{k+p})\right)\right]\right.$$

Numerical factors (indicated $=$) cancel:

$$= ie^2\int\frac{d^3k}{(2\pi)^3}\int dk^0\,\left(2k_\mu k_\nu - ((k^0)^2-\vec{k}^2)g_{\mu\nu}\right) \times \frac{n_F(|k^0|)}{\omega_k\omega_{k+p}}$$
$$\left\{\left(\delta(k^0-\omega_{k+p})+\delta(k^0+\omega_{k+p})\right)\left[\left(\frac{1}{k^0-\omega_k}-\frac{1}{k^0+\omega_k}\right)-i\pi\left(\delta(k^0-\omega_k)+\delta(k^0+\omega_k)\right)\right]\right.$$
$$\left.+\left(\delta(k^0-\omega_k)+\delta(k^0-\omega_k)\right)\left[\left(\frac{1}{k^0-\omega_{k+p}}-\frac{1}{k^0+\omega_{k+p}}\right)-i\pi\left(\delta(k^0-\omega_{k+p})+\delta(k^0+\omega_{k+p})\right)\right]\right\}$$

Look at $(\mu,\nu)=(0,0)$ component: $g_{00}=+1$

$$i\Pi_{00}^{HTL} = ie^2\int\frac{d^3k}{(2\pi)^3}\int dk^0\,\left(2k^0k^0-((k^0)^2-\vec{k}^2)g_{00}\right)\times\ldots$$
$$= ie^2\int\frac{d^3k}{(2\pi)^3}\int dk^0\left((k^0)^2+\vec{k}^2\right)\times\frac{n_F(|k^0|)}{\omega_k\omega_{k+p}}\left\{\ldots\right\}$$

Perform integration over k^0, using leading δ-function:

In first term, fix $k^0 \to \pm\omega_{k+p}$; in second term, fix $k^0 \to \pm\omega_k$.

$$
\begin{aligned}
i\Pi^{HTL}_{00} = ie^2\int\frac{d^3k}{(2\pi)^3}\Big\{&(\omega_{k+p}^2+\vec{k}^2)\frac{n_F(\omega_{k+p})}{\omega_k\omega_{k+p}}\Big[\Big(\frac{1}{\omega_{k+p}-\omega_k}-\frac{1}{\omega_{k+p}+\omega_k}\Big)-i\pi\Big(\delta(\omega_{k+p}-\omega_k)+\delta(\omega_{k+p}+\omega_k)\Big)\\
&+\Big(\frac{1}{-\omega_{k+p}-\omega_k}-\frac{1}{-\omega_{k+p}+\omega_k}\Big)-i\pi\Big(\delta(-\omega_{k+p}-\omega_k)+\delta(-\omega_{k+p}+\omega_k)\Big)\Big]\\
&+(\omega_k^2+\vec{k}^2)\frac{n_F(\omega_k)}{\omega_k\omega_{k+p}}\Big[\Big(\frac{1}{\omega_k-\omega_{k+p}}-\frac{1}{\omega_k+\omega_{k+p}}\Big)-i\pi\Big(\delta(\omega_k-\omega_{k+p})+\delta(\omega_k+\omega_{k+p})\Big)\\
&+\Big(\frac{1}{-\omega_k-\omega_{k+p}}-\frac{1}{-\omega_k+\omega_{k+p}}\Big)-i\pi\Big(\delta(-\omega_k-\omega_{k+p})+\delta(-\omega_k+\omega_{k+p})\Big)\Big]
\end{aligned}
$$

Use symmetry properties to add:

$$
\begin{aligned}
= 2ie^2\int\frac{d^3k}{(2\pi)^3}\Big\{&(\omega_{k+p}^2+\vec{k}^2)\frac{n_F(\omega_{k+p})}{\omega_k\omega_{k+p}}\Big[\overbrace{\frac{1}{\omega_{k+p}-\omega_k}-\frac{1}{\omega_{k+p}+\omega_k}}^{2\omega_k/(\omega_{k+p}^2-\omega_k^2)}-i\pi\Big(\delta(\omega_{k+p}-\omega_k)+\delta(\omega_{k+p}+\omega_k)\Big)\Big]\\
&+(\omega_k^2+\vec{k}^2)\frac{n_F(\omega_k)}{\omega_k\omega_{k+p}}\Big[\underbrace{\frac{1}{\omega_k-\omega_{k+p}}-\frac{1}{\omega_k+\omega_{k+p}}}_{-2\omega_{k+p}/(\omega_{k+p}^2-\omega_k^2)}-i\pi\Big(\delta(\omega_k-\omega_{k+p})+\delta(\omega_k+\omega_{k+p})\Big)\Big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
= 2ie^2\int\frac{d^3k}{(2\pi)^3}\Big\{&(\omega_{k+p}^2+\vec{k}^2)\frac{n_F(\omega_{k+p})}{\omega_k\omega_{k+p}}\Big[\frac{2\omega_k}{\omega_{k+p}^2-\omega_k^2}-i\pi\Big(\delta(\omega_{k+p}-\omega_k)+\delta(\omega_{k+p}+\omega_k)\Big)\Big]\\
&+(\omega_k^2+\vec{k}^2)\frac{n_F(\omega_k)}{\omega_k\omega_{k+p}}\Big[\frac{-2\omega_{k+p}}{\omega_{k+p}^2-\omega_k^2}-i\pi\Big(\delta(\omega_{k+p}-\omega_k)+\delta(\omega_{k+p}+\omega_k)\Big)\Big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
= 2ie^2\int\frac{d^3k}{(2\pi)^3}\Big[&\frac{2}{\omega_{k+p}^2-\omega_k^2}\Big(\frac{\omega_{k+p}^2+\vec{k}^2}{\omega_{k+p}}n_F(\omega_{k+p})-\frac{\omega_k^2+\vec{k}^2}{\omega_k}n_F(\omega_k)\Big)\\
&-\frac{i\pi}{\omega_k\omega_{k+p}}\Big((\omega_{k+p}^2+\vec{k}^2)n_F(\omega_{k+p})+(\omega_k^2+\vec{k}^2)n_F(\omega_k)\Big)\Big(\delta(\omega_{k+p}-\omega_k)+\underbrace{\delta(\omega_{k+p}+\omega_k)}_{\text{always vanishes because positive}}\Big)
\end{aligned}
$$

Write this as

$$\delta(\omega_{k+p}-\omega_k) = \frac{2\cos\theta}{|\vec{p}|^2}\,\delta\Big(|\vec{k}|+\frac{|\vec{p}|}{2\cos\theta}\Big)$$

look at first term:

$$= 4ie^2 \int \frac{d^3k}{(2\pi)^3} \left[\frac{1}{\omega_{k+p}^2 - \omega_k^2} \left(\frac{\omega_{k+p}^2 + |\vec{k}|^2}{\omega_{k+p}} n_F(\omega_{k+p}) - \frac{\omega_k^2 + |\vec{k}|^2}{\omega_k} n_F(\omega_k) \right) + \ldots \right]$$

$$\omega_{k+p} = \sqrt{|\vec{k}|^2 + |\vec{p}|^2 + 2|\vec{k}||\vec{p}|\cos\theta + m^2}$$

$$\omega_k = \sqrt{|\vec{k}|^2 + m^2}$$

$$n_F(\omega) = \frac{1}{e^{\beta\omega} + 1}$$

Expand as a series in $|\vec{p}| \approx 0$. Then expand as a series in $m^2 \approx 0$.

$$i\Pi_{00}^{HTL} \approx 4ie^2 \frac{1}{(2\pi)^3} \underbrace{\int d\Omega_3}_{4\pi} \int d|\vec{k}|\ |\vec{k}|^2 \left[-\frac{e^{|\vec{k}|/T}}{T(1+e^{|\vec{k}|/T})^2} + O(m^2, |\vec{p}|^2) \right]$$

Take $|\vec{k}| = Tx$

$$\approx 4ie^2 \frac{4\pi}{(2\pi)^3} \int d(Tx)\ (Tx)^2 \left(-\frac{e^x}{T(1+e^x)^2} \right)$$

$$= -4ie^2 \frac{4\pi}{(2\pi)^3} T^2 \underbrace{\int_0^\infty dx \frac{x^2 e^x}{(1+e^x)^2}}_{\pi^2/6}$$

$$= -4ie^2 \frac{4\pi}{(2\pi)^3} \frac{\pi^2}{6} T^2$$

$$\Pi_{00}^{HTL} = -\frac{e^2 T^2}{3}$$