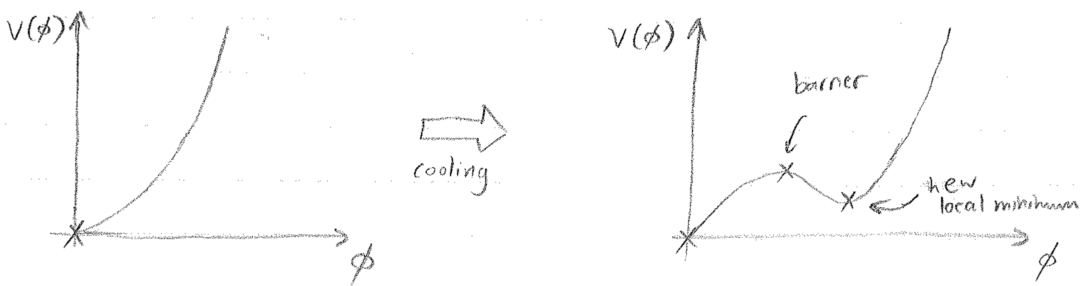
High Temperature Phase Transition Analysis

Finite temperature effective potential:
(Free energy density)

$e=0$ in SM (appears if there are cubic terms at tree-level)

T-dependence logarithmic

$$V(\phi, T) = D(T^2 - T_0^2)\phi^2 - \left(E - \frac{e}{T}\right)T\phi^3 + \frac{\lambda(T)}{4}\phi^4$$



(discriminant negative)
Initially (at high temperatures) symmetry is unbroken

(discriminant positive)
At temperature t_1, a new local minimum appears and a barrier is developed.

Find minima:

$$\frac{dV}{d\phi} = 2D(T^2 - T_0^2)\phi - 3\left(E - \frac{e}{T}\right)T\phi^2 + \lambda(T)\phi^4 = 0$$

$$2D(T^2 - T_0^2) - 3\left(E - \frac{e}{T}\right)T\phi + \lambda(T)\phi^3 = 0$$

three solutions: $\phi = 0, \quad \phi = \dfrac{+3\left(E - \frac{e}{T}\right)T \pm \sqrt{9\left(E - \frac{e}{T}\right)^2 T^2 - 8\lambda(T)D \times (T^2 - T_0^2)}}{2\lambda(T)}$

In order for symm. restoration, disc < 0 $\forall\, \alpha$ at high T. $\Rightarrow$ If $9E^2 > 8\lambda D$ for any α, symmetry is not restored.

$$9\left(E - \frac{e}{T_1}\right)^2 T_1^2 - 8\lambda(T_1)(T_1^2 - T_0^2) = 0$$

$$T_1 = \frac{2\sqrt{2}}{3}\,\frac{\sqrt{\lambda\left(T_0^2(8\lambda/9 - E^2) + e^2\right)}}{8\lambda/9 - E^2} - \frac{E}{8\lambda/9 - E^2}\,e$$

barrier at $\phi_{bar} = \dfrac{3\left(E - \frac{e}{T}\right)T}{2\lambda(T)} - \dfrac{\sqrt{9\left(E - \frac{e}{T}\right)^2 T^2 - 8\lambda(T)D \times (T^2 - T_0^2)}}{2\lambda(T)}$

minimum at $\phi_{\text{local min}} = \dfrac{3\left(E - \frac{e}{T}\right)T}{2\lambda(T)} + \dfrac{\sqrt{9\left(E - \frac{e}{T}\right)^2 T^2 - 8\lambda(T)D \times (T^2 - T_0^2)}}{2\lambda(T)}$

After further cooling, the potential at the minimum at $\phi = 0$ matches that at $\phi \neq 0$ (degerate)

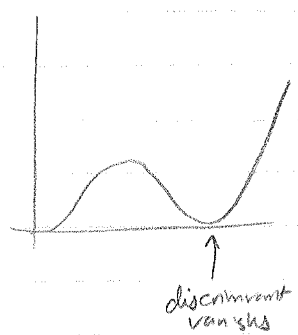
Plug $\phi_{\substack{local \\ min}}$ into $V(\phi, T)$ and set equal to $V(\phi = 0, T) = 0$

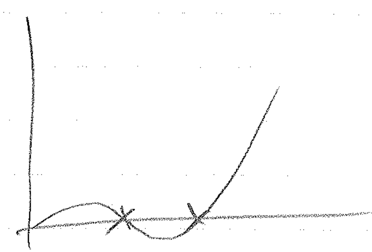
or easier:

Solve $V(\phi_c, T) = 0 = D(T^2 - T_0^2)\phi^2 - (E - \frac{e}{T})\phi^3 + \frac{\lambda(T)}{4}\phi^4$ for ϕ.

$$0 = D(T^2 - T_0^2) - (E - \frac{e}{T})T\phi + \frac{\lambda(T)}{4}\phi^2 \quad \text{(double root at } \phi = 0)$$

$$\phi = \frac{(E - \frac{e}{T})T \pm \sqrt{(E - \frac{e}{T})^2 T^2 - \lambda(T) D \times (T^2 - T_0^2)}}{2\lambda(T)/4}$$





$$\phi_{crit} = \frac{2(E - \frac{e}{T_c})T_c}{\lambda(T_c)} \Rightarrow \boxed{\frac{\phi_c}{T_c} = \frac{2E}{\lambda(T_c)} - \frac{2e}{T_c \lambda(T_c)}}$$

To find T_c, set Discriminant $= 0$

$$(E - \frac{e}{T_c})^2 T_c^2 - \lambda(T_c) D \times (T_c^2 - T_0^2) = 0$$

$$\boxed{\begin{aligned} T_c &= \text{sgn}(E)\left(\frac{\sqrt{D\lambda((D\lambda - E^2)T_0^2 + e^2)}}{D\lambda - E^2} - \frac{|E|e}{D\lambda - E^2}\right) \\ \phi_c &= \frac{2|E|\sqrt{D\lambda((D\lambda - E^2)T_0^2 + e^2)}}{\lambda(D\lambda - E^2)} - \frac{2eD}{D\lambda - E^2} \end{aligned}}$$

In SM (and other models without tree-level ϕ^3 terms in potential), $e = 0$.

Typically, E^2 is small compared to $D\lambda$.

Can approx:

$$T_c \approx T_0\left(1 + \frac{1}{2D\lambda}E^2 + \frac{3}{8}\frac{1}{D^2\lambda^2}E^4 + \dots\right)$$

— apparently, presence of E term raises critical temperature,

but makes the phase transition first order, with a nucleation temperature lower than T_c without E.