Charged Scalar Fields — Chemical Potentials

— with conserved charges.

$$\mathcal{L} = (\partial_\mu \Phi)^* (\partial^\mu \Phi) - m^2 \Phi^* \Phi$$

$$\hat{\Phi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(\hat{a}_{\vec{p}}\, e^{-ip.x} + \hat{b}^\dagger_{\vec{p}}\, e^{ip.x} \right)$$

Conserved charge: $\hat{Q} = \int d^3x\, \Phi^\dagger \overleftrightarrow{\partial_0} \Phi = \int \frac{d^3p}{(2\pi)^3} \left(\hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} - \hat{b}^\dagger_{\vec{p}} \hat{b}_{\vec{p}} \right)$

Box normalization

$$\hat{\Phi}(x) = \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k V}} \left(\hat{a}_{\vec{k}}\, e^{-ik.x} + \hat{b}^\dagger_{\vec{k}}\, e^{ik.x} \right)$$

$$\hat{Q} = \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \left(\hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} - \hat{b}^\dagger_{\vec{k}} \hat{b}_{\vec{k}} \right)$$

So, $Z(\beta, \mu) = \mathrm{Tr}\, e^{-\beta(\hat{H} - \mu \hat{Q})}$

Organize states: Modes are countable. Put all of them in a list $\{p_i\}$ & $\{k_j\}$, $i = 1,2,3\ldots$ and $j = 1,2,3,\ldots$ & identify each state with occupation numbers of each mode.

$\{p_i\}$ ↑ particles, $\{k_j\}$ ↑ antiparticles

Then $\hat{H} |n_1, n_2, \ldots; m_1, m_2 \ldots\rangle = \Big[\left(V\varepsilon_{vac} + (\omega_{p_1} n_1 + \omega_{p_2} n_2 + \ldots) \right) + \left(V\varepsilon_{vac} + (\omega_{p_1} m_1 + \omega_{p_2} m_2 + \ldots) \right) \Big] |n_1, n_2 \ldots; m_1, m_2 \ldots\rangle$

$$\hat{Q} |n_1, n_2, \ldots; m_1, m_2, \ldots\rangle = \left[n_1 + n_2 + n_3 + \ldots - m_1 - m_2 - m_3 - \ldots \right]$$

So, $Z(\beta,\mu) = \sum\limits_{\substack{n_1, n_2, \ldots \\ m_1, m_2, \ldots}} \langle n_1, n_2, \ldots | e^{-\beta(\hat{H} - \mu \hat{Q})} | n_1, n_2 \ldots\rangle$

$$= \sum_{n_1, n_2, \ldots} e^{-\beta\left(V\varepsilon_{vac} + \omega_{\vec{p}_1} n_1 + \ldots V\varepsilon_{vac} + \omega_{\vec{p}_1} m_1 + \ldots - \mu(n_1 + n_2 + \ldots - m_1 - m_2 - \ldots) \right)}$$

$$Z(\beta,\mu) = e^{-2\beta V \varepsilon_{vac}} \underbrace{\left(\sum_{n_1=0}^{\infty} e^{-\beta(\omega_{p_1}-\mu)n_1}\right)}_{\substack{\text{geometric series.} \\ =\frac{1}{1-e^{-\beta(\omega_{p_1}-\mu)}}}} \cdots \underbrace{\left(\sum_{m_1=0}^{\infty} e^{-\beta(\omega_{p_1}+\mu)m_1}\right)}_{\frac{1}{1-e^{-\beta(\omega_{p_1}+\mu)}}}$$

group by pairs

$$= e^{-2\beta V \varepsilon_{vac}} \prod_{\text{modes}, i} \left(\frac{1}{1-e^{-\beta(\omega_{p_i}-\mu)}} \, \frac{1}{1-e^{-\beta(\omega_{p_i}+\mu)}} \right)$$

Grand Potential: (analog of Free energy) $\quad -N = \left.\frac{\partial F_{Grand}}{\partial \mu}\right|_{T,V}$

$$F_{Grand} = \frac{-1}{\beta} \ln Z(\beta,\mu)$$

$$= -\frac{1}{\beta}\left[-2\beta V \varepsilon_{vac} + \sum_i \ln\left(\frac{1}{1-e^{-\beta(\omega_i-\mu)}} \, \frac{1}{1-e^{-\beta(\omega_i+\mu)}} \right)\right]$$

$$= 2V\varepsilon_{vac} + \frac{V}{\beta}\int \frac{d^3p}{(2\pi)^3} \ln\left(1-e^{-\beta(\omega_{\vec p}-\mu)}\right) + \ln\left(1-e^{-\beta(\omega_{\vec p}+\mu)}\right)$$

$$= 2V\varepsilon_{vac} + \frac{V}{\beta}\int \frac{d\Omega_3}{(2\pi)^3} \int dp \left[p^2 \ln\left(1-e^{-\beta(\omega_p-\mu)}\right) + p^2 \ln\left(1-e^{-\beta(\omega_p+\mu)}\right) \right]$$

Now write for small μ:

$$\ln\left(1-e^{-\beta(\omega_p-\mu)}\right) \simeq \ln\left(1-e^{-\beta\omega_p}\right) + \mu \frac{-\beta e^{-\beta\omega_p}}{1-e^{-\beta\omega_p}} + \frac{\mu^2}{2}\frac{-\beta^2 e^{\beta\omega}}{(e^{\beta\omega}-1)^2} + \ldots$$

$$= \ln\left(1-e^{-\beta\omega_p}\right) - \mu\beta \frac{1}{e^{\beta\omega_p}-1} \qquad \text{assume } \omega_p = p \text{ (massless)}$$

$$F_{Grand} = 2V\varepsilon_{vac} + \frac{V}{\beta}\int \overset{4\pi}{\overbrace{\frac{d\Omega_3}{(2\pi)^3}}} \left[\frac{2J_B}{\beta^3} - \mu^2\beta^2 \underbrace{\int dp \; p^2 \frac{e^{\beta p}}{(e^{\beta p}-1)^2}}_{\frac{1}{\beta^3}\frac{\pi^2}{3}} \right]$$

$$F_{Grand} \simeq 2V\varepsilon_{vac} + \frac{V}{\beta}\frac{1}{2\pi^2}\left[\frac{2J_B}{\beta^3} - \frac{\mu^2}{\beta}\frac{\pi^2}{3} + O(\mu^3) \right]$$

$$-\langle Q \rangle = \frac{\partial F}{\partial \mu} = \frac{\partial}{\partial \mu} \frac{V}{\beta}\frac{1}{2\pi^2}\frac{-\mu^2}{\beta}\frac{\pi^2}{3} = \frac{-V}{\cancel{6}_{3}\beta^2} 2\mu = \frac{-V\mu}{3\beta^2}$$

For fermions, we have spin-up and spin-down. So, states are labeled with an additional quantum #:

$$\hat{H}\,|n_{1\uparrow}, n_{1\downarrow}, n_{2\uparrow}, n_{2\downarrow}, \ldots ; m_{1\uparrow}, m_{1\downarrow}, m_{2\uparrow}, m_{2\downarrow}, \ldots\rangle$$
$$= \left[-2V\mathcal{E}_{vac} + (\omega_{p_1} n_{1\uparrow} + \omega_{p_1} n_{1\downarrow} + \ldots) - 2V\mathcal{E}_{vac} + (\omega_{p_1} m_{1\uparrow} + \omega_{p_1} m_{1\downarrow} + \ldots)\right]$$

Similarly,

$$\hat{Q}\,|n_{1\uparrow}, n_{1\downarrow} \ldots ; m_{1\uparrow}, m_{1\downarrow}, \ldots\rangle = \left[n_{1\uparrow} + n_{1\downarrow} + \ldots - m_{1\uparrow} - m_{1\downarrow} - \ldots\right]$$

So,

$$Z(\beta,\mu) = \sum_{\substack{n_{1\uparrow}, n_{1\downarrow}, \ldots \\ m_{1\uparrow}, m_{1\downarrow}, \ldots}}^{1} \langle n_{1\uparrow}, \ldots |\, e^{-\beta(\hat{H} - \mu\hat{Q})}\, | n_{1\uparrow}, \ldots\rangle$$

$$= \sum_{n_{1\uparrow}, n_{1\downarrow}, \ldots}^{1} e^{-\beta\left(-2V\mathcal{E}_{vac} + \omega_{\vec{p}_1} n_{1\uparrow} \ldots - 2V\mathcal{E}_{vac} + \omega_{\vec{p}_1} m_{1\uparrow} - \mu(n_{1\uparrow} + \ldots m_{1\uparrow} + \ldots)\right)}$$

$$= e^{4\beta V\mathcal{E}_{vac}} \left(\sum_{n_{1\uparrow}=0}^{1} e^{-\beta(\omega_{\vec{p}_1} - \mu) n_{1\uparrow}}\right) \ldots \left(\sum_{m_{1\uparrow}=0}^{1} e^{-\beta(\omega_{\vec{p}_1} + \mu) m_{1\uparrow}}\right) \ldots$$

$$= e^{4\beta V\mathcal{E}_{vac}} \prod_{\text{modes}, i} \left(1 + e^{-\beta(\omega_{p_i} - \mu)}\right)^2 \left(1 + e^{-\beta(\omega_{p_i} + \mu)}\right)^{2 \leftarrow \text{spin DoF}}$$

<u>Grand potential</u>

$$F_{\text{Grand}} = \frac{-1}{\beta} \ln Z(\beta,\mu)$$

$$= \frac{-1}{\cancel{\beta}} 4\cancel{\beta} V\mathcal{E}_{vac} - \frac{1}{\beta}\, 2\, V \int \frac{d^3p}{(2\pi)^3} \left[\ln\left(1 + e^{-\beta(\omega_p - \mu)}\right) + \ln\left(1 + e^{-\beta(\omega_p + \mu)}\right)\right]$$

For small μ,

$$\ln\left(1 + e^{-\beta(\omega_p \pm \mu)}\right) = \ln\left(1 + e^{-\beta\omega_p}\right) \mp \mu\beta \frac{e^{-\beta\omega}}{1 + e^{-\beta\omega}} + \frac{\mu^2}{2}\beta^2 \frac{e^{\beta\omega}}{(1 + e^{\beta\omega})^2}$$

assume massless fermions: $\omega_p \sim |\vec{p}|$.

$$F_{\text{Grand}} = -4V\mathcal{E}_{vac} - \frac{2V}{\beta}\int \frac{\overbrace{d\Omega_3}^{4\pi}}{(2\pi)^3}\int_0^\infty \underbrace{dp\, p^2}_{\text{rescale } p \to p/\beta}\left[2\ln(1+e^{-\beta p}) + \mu^2\beta^2\frac{e^{\beta p}}{(1+e^{\beta p})^2}\right]$$

$$= -4V\mathcal{E}_{vac} - \frac{2V}{\beta^4}\frac{1}{2\pi^2}\left(2J_F(0) + \mu^2\beta^2\underbrace{\int_0^\infty p^2\frac{e^p}{(1+e^p)^2}}_{\pi^2/6}\right)$$

$$= -4V\mathcal{E}_{vac} - \frac{V}{\pi^2\beta^4}J_F(0) - \frac{\cancel{2}V}{\cancel{\beta^4}\beta^2}\frac{1}{\cancel{2\pi^2}}\mu^2\cancel{\beta^2}\frac{\cancel{\pi^2}}{6}$$

$$= -4V\mathcal{E}_{vac} - \frac{V}{\pi^2\beta^4}J_F(0) - \mu^2\frac{V}{6\beta^2}$$

$$\Rightarrow \langle Q\rangle = -\frac{\partial F_{\text{Grand}}}{\partial\mu} = -\left(-\cancel{2}\mu\frac{V}{\underset{3}{\cancel{6}}\beta^2}\right) = \frac{V}{3\beta^2}\mu$$

solving for $\mu = 3\beta^2\frac{\langle Q\rangle}{V}$

In SM: If Q = Baryon number, we must take color and weak isospin DOF into account. Also 1 quark DOF = 1/3 Baryon DOF

$$Z(\beta,\mu) = e^{\overbrace{(\ )}^{2\times N_c\times 2}\beta V\mathcal{E}_{vac}}\prod_{\text{modes},i}\left(1+e^{-\beta(\omega_{p_i}-\frac{1}{3}\mu)}\right)^{2\times N_c\times \overbrace{2}^{\text{weak isospin}}\times n_f}\left(1+e^{-\beta(\omega_{p_i}+\frac{\mu}{3})}\right)^{2\times N_c\times 2\times n_f}$$

$$\Rightarrow F_{\text{Grand}} = -(2\times N_c\times 2n_f)V\mathcal{E}_{vac} - \frac{(2\times N_c\times 2\times n_f)}{2}\frac{V}{\pi^2\beta^4}J_F(0) + \frac{\mu^2}{9}\frac{(2\times N_c\times 2\times n_f)}{2}\frac{V}{6\beta^2}$$

$$\text{So}\quad \langle N_B\rangle = \frac{(2\times \cancel{N_c}\times 2\times n_f)V}{\cancel{2}}\frac{1}{\cancel{6}\beta^2}\frac{\cancel{2}\mu}{9} = \frac{2Vn_f}{9\beta^2}\mu \quad\text{or}\quad \mu = \frac{9}{2n_f}\beta^2\frac{\langle N_B\rangle}{V}$$

If Q = Lepton number, e_L, e_R, ν_L $\quad$ $\bar{e}_L, \bar{e}_R$ & $\bar{\nu}_R$ (helicities)

$$Z(\beta,\mu) = e^{(\ )\beta V\mathcal{E}_{vac}}\prod_{\text{modes},i}\left(1+e^{-\beta(\omega_{p_i}-\mu)}\right)^{3\times n_f}\left(1+e^{-\beta(\omega_{p_i}-\frac{\mu}{3})}\right)^{3\times n_f}$$

$$F = \ldots\ldots + \mu^2\frac{(2\times 3\times n_f)}{2}\frac{V}{6\beta^2}$$

$$\text{So}\quad \langle N_L\rangle = \frac{(3\times n_f)}{\cancel{2}}\frac{V}{6\beta^2}\cancel{2}\mu = \frac{Vn_f}{2\beta^2}\mu \quad\text{or}\quad \mu = \frac{2}{n_f}\beta^2\frac{\langle N_L\rangle}{V}$$