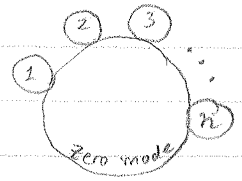
Performing the ring-sum — ϕ^4 theory.

$\phi_0 \equiv$ Matsubara zero mode.
$\phi \equiv$ all other modes.



$$\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4$$

$$\xrightarrow{\phi=\phi_0+\phi} -\frac{\lambda}{4!}\left(\phi^4 + 4\phi_0\phi^3 + \underbrace{6\phi_0^2\phi^2}_{\text{relevant interaction.}} + 4\phi_0^3\phi + \phi_0^4\right)$$

$$V_{eff}^{(n)} = \ldots \underbrace{-\hbar}_{\text{Def}^{n}\ V_{eff}}\ \underbrace{(n-1)!\,\frac{2^n}{2}}_{\text{Building the ring}}\ \underbrace{\frac{1}{n!}}_{\text{Taylor}}\ \underbrace{\int\frac{d^3k}{(2\pi)^3}\left(\frac{\hbar}{\vec{k}^2+m^2(\phi)}\right)^n}_{\text{Zero mode loop integral}}\ \underbrace{\left[\frac{-6\lambda}{4!\hbar}\frac{1}{\beta}\sum_n\int\frac{d^3p}{(2\pi)^3}\frac{\hbar}{\omega_n^2+\vec{p}^2+m^2(\phi)}\right]^n}_{\text{Daisy petals}}$$

$$= -\lambda T^2/48 + O(T)\ldots$$

$$= -\hbar\,\frac{T}{2}\,\frac{1}{n}\int\frac{d^3k}{(2\pi)^3}\left(\frac{\hbar}{\vec{k}^2+m^2(\phi)}\right)^n\left(\frac{-\lambda T^2}{24}\right)^n$$

So, the full contribution (add up all n-petal rings) is:

$$V_{ring} = -\hbar\,\frac{T}{2}\int\frac{d^3k}{(2\pi)^3}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\hbar}{\vec{k}^2+m^2(\phi)}\right)^n\left(\frac{-\lambda T^2}{24}\right)^n$$

$$= -\hbar\,\frac{T}{2}\int\frac{d^3k}{(2\pi)^3}\,-\ln\left(1+\frac{-\hbar\lambda T^2/24}{\vec{k}^2+m^2(\phi)}\right)$$

$$= +\hbar\,\frac{T}{2}\int\frac{d^3k}{(2\pi)^3}\,\ln\left(\vec{k}^2+m^2(\phi)-\hbar\frac{\lambda T^2}{24}\right)-\ln\left(\vec{k}^2-m^2(\phi)\right)$$

To integrate: $\displaystyle\int\frac{d^3k}{(2\pi)^3}\ln\left(\vec{k}^2+\bar{m}^2\right)$

$$= \int d\bar{m}^2\int\frac{d^3k}{(2\pi)^3}\frac{1}{k^2+\bar{m}^2} \qquad \text{Dim reg:}$$

$$= \int d\bar{m}^2\,\frac{1}{(2\pi)^3}\,\frac{2\pi^{3/2}}{\Gamma(3/2)}\,(\bar{m}^2)^{1/2}\,\frac{\Gamma(3/2)}{2}\,\frac{\Gamma(-1/2)}{0!}$$

$$= \frac{-1}{4\pi}\int d\bar{m}^2\,(\bar{m}^2)^{1/2}$$

$$= \frac{-1}{4\pi}\,\frac{2}{3}\,(\bar{m}^2)^{3/2} = \frac{-1}{6\pi}(\bar{m})^{3/2}$$

$$= -\hbar\,\frac{T}{12\pi}\left[\left(m^2(\phi)+\hbar\frac{\lambda T^2}{24}\right)^{3/2}-\left(m^2(\phi)\right)^{3/2}\right]$$

$$m^2(\phi) = -\mu^2+\frac{\lambda\phi^2}{2}$$