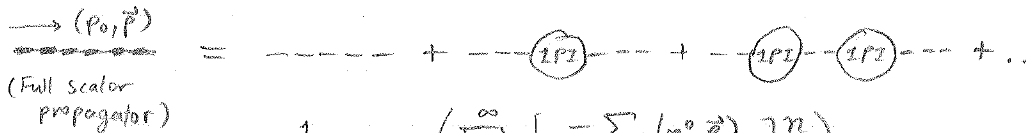
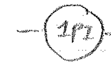
ϕ^4 One-loop Mass Correction (Finite temperature, TADPOLE GRAPH)

Setup:



$$= \frac{1}{p_0^2 + \vec{p}^{\,2} + m^2} \left(\sum_{n=0}^{\infty} \left[\frac{-\Sigma_\beta(p^0, \vec{p})}{p_0^2 + \vec{p}^{\,2} + m^2} \right]^n \right)$$

$$= \frac{1}{p_0^2 + \vec{p}^{\,2} + m^2 + \Sigma_\beta(p^0, \vec{p})} ,$$



$$= -\Sigma_\beta(p^0, \vec{p})$$

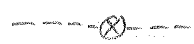
Hence $\Delta m^2_{pole} = \Sigma_\beta(p^0 = 0, \vec{p}^{\,2} = 0)$

Feynman Rules:

$$= \frac{1}{\omega_n^2 + \vec{p}^{\,2} + m_R^2}$$

$$= -\frac{\lambda_R \mu^{2\epsilon}}{4!}$$

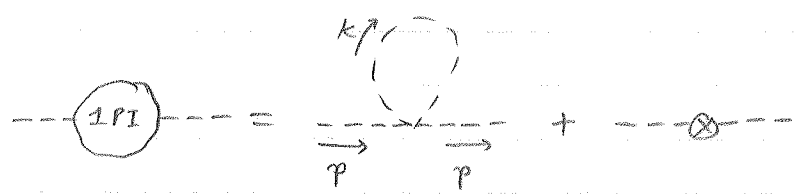
"renormalized"



$$= -\delta_{m^2}/2$$

So, to one-loop, the self-energy correction at finite temperature is:

$$-\Sigma_\beta(p_0, \vec{p}) =$$



$$= \underbrace{4 \times 3}_{\text{contractions}} \left(\frac{-\lambda \mu^{2\epsilon}}{4!} \right) \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{1}{\omega_n^2 + \vec{k}^2 + m^2} + \underset{\text{contractions}}{2} \left(\frac{-\delta_{m^2}}{2} \right)$$

$\omega_n = \frac{2\pi n}{\beta}$

No external momentum $(p^0, \vec{p})$ dependence — immediately take $p_0 = \vec{p} = 0$.

$$+\Delta m^2_{pole} = \frac{+\lambda}{2\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\frac{4\pi^2 n^2}{\beta^2} + \vec{k}^2 + m^2} + \delta_{m^2}$$

$$= \frac{\lambda}{2\beta} \left(\frac{\beta^2}{4\pi^2} \right) \sum_{n=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3} \frac{1}{n^2 + \frac{\beta^2 \omega_k^2}{4\pi^2}} , \qquad \omega_k^2 = \vec{k}^2 + m^2$$

Use $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{n^2+y^2} = \frac{\pi}{y}\coth(\pi y)$ (method of residues)

$$\Delta m^2_{pole} = \frac{\lambda}{2\beta}\left(\frac{\beta^2}{4\pi^2}\right)\int\frac{d^3k}{(2\pi)^3}\left(\frac{\pi}{\beta\omega_k/4\pi}\right)\coth\left(\pi\frac{\beta\omega_k}{4\pi}\right) + \delta_{m^2}$$

$$= \frac{\lambda}{4}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k}\coth\left(\frac{\beta\omega_k}{2}\right) + \delta_{m^2}$$

Now write $\coth\left(\frac{\beta\omega_k}{2}\right) = 1 + 2n_B(\omega_k) = 1 + 2\left(\frac{1}{e^{\beta\omega_k}-1}\right)$

$$\Delta m^2_{pole} = \frac{\lambda}{4}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k}\left(1 + \frac{2}{e^{\beta\omega_k}-1}\right) + \delta_{m^2}$$

$$= \underbrace{\frac{\lambda}{4}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k} + \delta_{m^2}}_{\substack{\text{Temperature independent contribution} \\ \text{— Due to quantum fluctuations} \\ \text{(zero temp. } \Sigma\text{)}}} + \underbrace{\frac{\lambda}{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k}\frac{1}{e^{\beta\omega_k}-1}}_{\substack{\text{Contribution due to finite} \\ \text{temperature: } = \Delta m_T^2}}$$

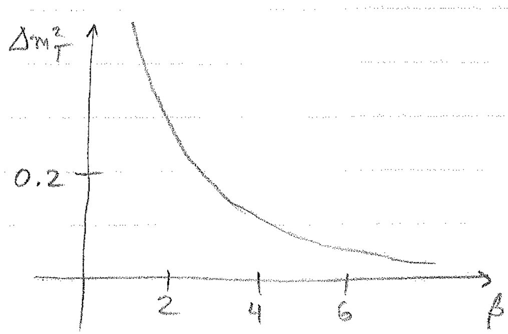
$$\Delta m_T^2 = \frac{\lambda}{(2\pi)^3}\frac{1}{2}\underbrace{\int d\Omega_3}_{4\pi}\int d|\vec{k}|\frac{1}{\sqrt{|\vec{k}|^2+m^2}}\frac{|\vec{k}|^2}{\left(e^{\beta\sqrt{|\vec{k}|^2+m^2}}-1\right)}$$

rescale: $\vec{k} \to \frac{x}{\beta}$ (ch. of integration variables)

$$= \frac{4\pi\lambda}{(2\pi)^3}\frac{1}{\beta^2}\frac{1}{2}\int_0^\infty dx\frac{x^2}{\sqrt{x^2+m^2\beta^2}\left(e^{\sqrt{x^2+m^2\beta^2}}-1\right)}$$

$$= \frac{\lambda}{2\pi^2\beta^2}\frac{\partial}{\partial z^2}J_B(z^2)\Big|_{z^2=m^2\beta^2} \sim \frac{\lambda}{2\pi^2\beta^2}\left(\frac{\pi^2}{12}+\ldots\right) = \frac{\lambda}{24\beta^2}+\ldots$$

(see high temperature expansion)



Notice that for low β (high temperatures) (equivalently low mass),

Δm_T^2 grows very <u>large</u>! One-loop expansion breaks down.

P. Arnold, O. Espinosa.
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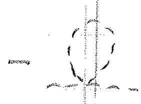
(Matsubara) Mode Decomposition of $\Sigma_\beta(p^0,\vec{p})$ (one-loop ϕ^4 theory)

$(d=4-2\epsilon)$



$$= \Delta m^2_{pole} = \frac{\lambda\mu^{2\epsilon}}{2\beta}\sum_{n=-\infty}^{+\infty}\int\frac{d^{d-1}k}{(2\pi)^{d-1}}\frac{1}{\left(\frac{4\pi^2n^2}{\beta^2}\right)+\vec{k}^2+m^2} + \delta_{m^2}$$

Split into zero $(n=0)$ mode and heavy $(n\neq 0)$ modes.

$$\Delta m^2_{pole} = \underbrace{\frac{\lambda\mu^{2\epsilon}}{2\beta}\int\frac{d^{d-1}k}{(2\pi)^{d-1}}\frac{1}{\vec{k}^2+m^2}}_{\text{zero mode}} + \frac{\lambda\mu^{2\epsilon}}{2\beta}\,2\sum_{n=1}^{\infty}\int\frac{d^{d-1}k}{(2\pi)^{d-1}}\frac{1}{\left(\frac{4\pi^2n^2}{\beta^2}\right)+\vec{k}^2+m^2} + \delta_{m^2}$$

Evaluate <u>zero mode part</u>

$$\begin{pmatrix}\text{zero}\\ \text{Mode}\end{pmatrix} = \frac{\lambda}{2\beta}\mu^{2\epsilon}\frac{1}{(2\pi)^{d-1}}\underbrace{\int d\Omega_{d-1}}_{\frac{2(\sqrt{\pi})^{d-1}}{\Gamma\left(\frac{d-1}{2}\right)}}\underbrace{\int_0^\infty d|\vec{k}|\,\frac{|\vec{k}|^{d-2}}{|\vec{k}|^2+m^2}}_{\frac{m^{d-3}}{2}\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{3-d}{2}\right)}$$

$$= -\frac{1}{8\pi}\frac{\lambda}{\beta}(m^2)^{1/2}$$ (finite as $d\to 4$, but nonanalytic in m^2)
(see pg. 3 for details)

Evaluate <u>heavy mode parts</u>:

Write $$\frac{1}{\left(\frac{4\pi^2n^2}{\beta^2}\right)+\vec{k}^2+m^2} = \sum_{l=0}^{\infty}\frac{(-1)^l(m^2)^l}{\left[\left(\frac{4\pi^2n^2}{\beta^2}\right)+\vec{k}^2\right]^{l+1}}$$ (as an expansion in m

$$\begin{pmatrix}\text{heavy}\\ \text{modes}\end{pmatrix} = \frac{\lambda\mu^{2\epsilon}}{2\beta}\,2\sum_{n=1}^{\infty}\sum_{l=0}^{\infty}(-1)^l(m^2)^l\underbrace{\int\frac{d^{d-1}k}{(2\pi)^{d-1}}\frac{1}{\left[\left(\frac{4\pi^2n^2}{\beta^2}\right)+\vec{k}^2\right]^{l+1}}} + \delta_{m^2}$$

$$\frac{1}{(2\pi)^{d-1}}\int d\Omega_{d-1}\int d|\vec{k}|\,\frac{1}{\left[\left(\frac{4\pi^2n^2}{\beta^2}\right)+|\vec{k}|^2\right]^{l+1}}$$

Gradshteyn. (1994)
p. 343 eqn 3.251(

$$= \frac{1}{(2\pi)^{d-1}}\frac{2\pi^{(d-1)/2}}{\cancel{\Gamma\left(\frac{d-1}{2}\right)}}\left(\frac{2\pi n}{\beta}\right)^{d-2l-3}\cdot\frac{1}{2}\frac{\cancel{\Gamma\left(\frac{d-1}{2}\right)}\Gamma\left(\frac{3-d}{2}+l\right)}{\underbrace{\Gamma(l+1)}_{=\,l!}}$$

Perform sum over n: $\sum_{n=1}^{\infty} n^{d-2l-3} = \zeta(2l-d+3)$

$$= \frac{\lambda\mu^{2\epsilon}}{2\beta}\,2\sum_{l=0}^{\infty}(-1)^l(m^2)^l\underbrace{\frac{2\pi^{(3-2\epsilon)/2}}{(2\pi)^{3-2\epsilon}}\left(\frac{2\pi}{\beta}\right)^{1-2l-2\epsilon}}_{\frac{1}{2\sqrt{\pi}\beta}\left(\frac{\beta^2}{\pi}\right)^{\epsilon}\left(\frac{\beta}{2\pi}\right)^{2l}\times\frac{1}{2}}\zeta(2l-1+2\epsilon)\frac{1}{2}\frac{\Gamma\left(l-\frac{1}{2}+\epsilon\right)}{l!} + \delta_{m^2}$$

$$\begin{pmatrix}\text{heavy}\\ \text{modes}\end{pmatrix} = \frac{\lambda\mu^{2\epsilon}}{2\beta}\,\frac{1}{2\sqrt{\pi}\,\beta}\left(\frac{\beta^2}{\pi}\right)^{\epsilon}\frac{1}{2}\cdot 2\sum_{l=0}^{\infty}(-1)^l (m^2)^l\left(\frac{\beta}{2\pi}\right)^{2l}\zeta(2l-1+2\epsilon)\,\frac{\Gamma(l-\frac{1}{2}+\epsilon)}{l!} + \delta_{m^2}$$

$$= \frac{\lambda}{4\sqrt{\pi}\beta^2}\left(\frac{\beta^2\mu^2}{\pi}\right)^{\epsilon}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\,\Gamma(l-\tfrac{1}{2}+\epsilon)\,\zeta(2l-1+2\epsilon)\left(\frac{\beta^2 m^2}{4\pi^2}\right)^{l}$$

Pull out $l=0$ and $l=1$ terms.

$$= \frac{\lambda}{4\sqrt{\pi}\beta^2}\left(\frac{\beta^2\mu^2}{\pi}\right)^{\epsilon}\Big\{\underbrace{\Gamma(-\tfrac{1}{2}+\epsilon)}_{-2\sqrt{\pi}}\,\underbrace{\zeta(-1+2\epsilon)}_{-1/12} \quad \longleftarrow l=0 \text{ term (finite } \epsilon\to 0)$$

$$+ (-1)\,\Gamma(1-\tfrac{1}{2}+\epsilon)\,\underbrace{\zeta(1+2\epsilon)}_{\text{divergent}}\left(\frac{\beta^2 m^2}{4\pi^2}\right) + \delta_{m^2} \quad \longleftarrow l=1 \text{ term (divergent } \epsilon\to 0)$$

$$+ \sum_{l=2}^{\infty}\frac{(-1)^l}{l!}\,\Gamma(l-\tfrac{1}{2}+\epsilon)\,\zeta(2l-1+2\epsilon)\left(\frac{\beta^2 m^2}{4\pi^2}\right)^{l}\Big\} \quad \longleftarrow l\geq 2 \text{ terms (finite } \epsilon\to 0)$$

$$= \underbrace{\frac{\lambda}{24\beta^2}}_{l=0}\ \underbrace{-\frac{\lambda}{32\pi^2}m^2\left(\frac{1}{\epsilon} - \gamma_E + \ln(4\pi) - \ln\left(\frac{T^2}{\mu^2}\right) - 2(\ln 4\pi - \gamma_E)\right)}_{l=1 \text{ (UV divergent)}} + \delta_{m^2}$$

$$+ \frac{\lambda}{4\sqrt{\pi}\beta^2}\sum_{l=2}^{\infty}\frac{(-1)^l}{l!}\,\Gamma(l-\tfrac{1}{2})\,\zeta(2l-1)\left(\frac{\beta^2 m^2}{4\pi^2}\right)^{l}$$

Therefore, having identified where each term in the high temperature expansion comes from, we can put everything together:

$$\Delta m^2_{\text{pole}} = \frac{\lambda}{24\beta^2}\ \underbrace{-\frac{1}{8\pi}\frac{\lambda}{\beta}(m^2)^{1/2}}_{\text{comes from zero mode}} - \frac{\lambda}{32\pi^2}m^2\left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi - \ln\left(\frac{T^2}{\mu^2}\right) - 2(\ln 4\pi - \gamma_E)\right)$$

$$+ \frac{\lambda}{4\sqrt{\pi}\beta^2}\sum_{l=2}^{\infty}\frac{(-1)^l}{l!}\,\Gamma(l-\tfrac{1}{2})\,\zeta(2l-1)\left(\frac{\beta^2 m^2}{4\pi^2}\right)^{l} + \delta_{m^2}$$

Zero-temperature counterterm to renormalize m^2.

Appendix

ZERO MODE integration: $\boxed{\text{— finite in Dimensional regularization}}$

$$\int_0^\infty dp \, \frac{p^{d-2}}{(p^2+m^2)^N}$$

integration will be independent of this choice — choose $\alpha = 1/2$ for convenience.

ch. var: $\quad p = m\left(\frac{1-x}{x}\right)^\alpha = m\, x^{-\alpha}(1-x)^\alpha$

Integration range:
$p: \; 0 \to \infty$
$x: \; 1 \to 0$

$$\Rightarrow dp = -\alpha m \, x^{-\alpha-1}(1-x)^{\alpha-1}$$

$$\int_1^0 dx \; -\alpha m \, x^{-\alpha-1}(1-x)^{\alpha-1} \frac{m^{d-2} x^{-\alpha(d-2)}(1-x)^{\alpha(d-2)}}{\left[m^2 x^{-2\alpha}(1-x)^{2\alpha} + m^2\right]^N}$$

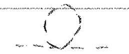
$$= +\alpha m^{d-2-N} \int_0^1 dx \; x^{-\alpha-1}(1-x)^{\alpha-1}\left[x^{-\alpha(d-2)}(1-x)^{\alpha(d-2)}\right]\frac{1}{\left[x^{-2\alpha}(1-x)^{2\alpha}+1\right]^N}$$

$$= \alpha m^{d-2-N}\int_0^1 dx \; x^{-\alpha-1}(1-x)^{\alpha-1}\left[x^{-\alpha(d-2)}(1-x)^{\alpha(d-2)}\right]\left[\frac{x^{2\alpha}}{(1-x)^{2\alpha}+x^{2\alpha}}\right]^N$$

$$= \alpha m^{d-2-N}\int_0^1 dx \; x^{-1-(d-1-2N)\alpha}(1-x)^{-1+(d-1)\alpha}\left[\frac{1}{(1-x)^{2\alpha}+x^{2\alpha}}\right]^N$$

At this point, convenient to choose $\alpha = 1/2$ — helps to combine terms in denominator.

$$= \frac{m^{d-2-N}}{2}\int_0^1 dx \; x^{\frac{2N-d-1}{2}}(1-x)^{\frac{d-3}{2}} = \frac{m^{d-3}}{2}\,\frac{\Gamma\left(\frac{2N-d+1}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}{\Gamma(N)}$$

For tadpole graph, , $N=1$ (one propagator)