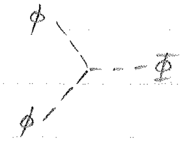
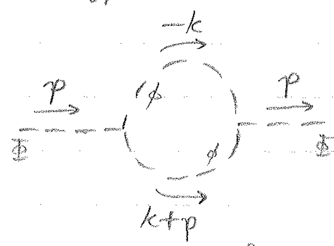
One-loop Self Energy in $g\phi^3$ Theory (Finite temperature BUBBLE GRAPH)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{2}\partial_\mu\Phi\,\partial^\mu\Phi - \frac{1}{2}M^2\phi^2 - \frac{g}{2}\Phi\phi\phi$$

Feynman Rule 

$= \frac{-g}{2}$

To one-loop, the self energy correction to Φ at finite temperature is:

$-\Sigma_\beta(p^0,\vec{p}) =$ 

contractions:

$$\langle 0|\Phi \quad \Phi\phi\phi \quad \Phi\phi\phi \quad \Phi|0\rangle = 2\times 2$$

$$= \underbrace{4\,\frac{1}{2!}\left(\frac{-g}{2}\right)^2\frac{1}{\beta}}_{g^2/2\beta}\sum_n\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{\left(\frac{2\pi n}{\beta}\right)^2+\underbrace{(-\vec{k})^2+m^2}_{=\omega_k^2}}\;\frac{1}{\left(\frac{2\pi n}{\beta}+\underset{\substack{2N\pi/\beta \\ \text{(external mode)}}}{p^0}\right)^2+\underbrace{(\vec{k}+\vec{p})^2+m^2}_{=\omega^2_{k+p}}}$$

Factor out $\left(\frac{2\pi}{\beta}\right)^{-2}\left(\frac{2\pi}{\beta}\right)^{-2}$ (one from each prop.)

$$= \frac{g^2}{2\beta}\left(\frac{\beta}{2\pi}\right)^4\sum_{n=-\infty}^{\infty}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{n^2+\left(\frac{\beta\omega_k}{2\pi}\right)^2}\;\frac{1}{\left(n+\frac{\beta p^0}{2\pi}\right)^2+\left(\frac{\beta\omega_{k+p}}{2\pi}\right)^2}$$

Then, use $\frac{1}{a^2+b^2} = \frac{i}{2b}\left(\frac{1}{a+ib} - \frac{1}{a-ib}\right)$ on each propagator.

$$= \frac{g^2}{2\beta}\left(\frac{\beta}{2\pi}\right)^{\cancel{4}\,2}\sum_{n=-\infty}^{\infty}\int\frac{d^3k}{(2\pi)^3}\,\frac{i\times i}{2\left(\frac{\cancel{\beta}\omega_k}{\cancel{2\pi}}\right)\left(\frac{\cancel{\beta}\omega_{k+p}}{\cancel{2\pi}}\right)}\left(\frac{1}{n+i\frac{\beta\omega_k}{2\pi}} - \frac{1}{n-i\frac{\beta\omega_k}{2\pi}}\right)$$

$$\times\left(\frac{1}{n+\frac{\beta p^0}{2\pi}+i\frac{\beta\omega_{k+p}}{2\pi}} - \frac{1}{n+\frac{\beta p^0}{2\pi}-i\frac{\beta\omega_{k+p}}{2\pi}}\right)$$

$$= \frac{g^2}{2\beta}\left(\frac{\beta}{2\pi}\right)^2\sum_{n=-\infty}^{\infty}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{4\omega_k\omega_{k+p}}\left(\frac{1}{n+i\frac{\beta\omega_k}{2\pi}} - \frac{1}{n-i\frac{\beta\omega_k}{2\pi}}\right)\left(\frac{1}{n+i\frac{\beta}{2\pi}(-ip^0+\omega_{k+p})} - \frac{1}{n+i\frac{\beta}{2\pi}(-ip^0-\omega_{k+p})}\right)$$

Expand product to give four terms of the form $\frac{1}{n+ix}\,\frac{1}{n+iy}$.

Frequency sum: $\displaystyle\sum_n \frac{1}{n+ix}\,\frac{1}{n+iy} = \frac{\pi}{x-y}\left[\coth(\pi x) - \coth(\pi y)\right].$

The four terms are:

1: $$\frac{1}{n+i\frac{\beta\omega_k}{2\pi}}\,\frac{1}{n+i\frac{\beta}{2\pi}(-ip^0+\omega_{k+p})} \xrightarrow{\text{sum}} \frac{2\pi^2}{\beta(\omega_k+ip^0-\omega_{k+p})}\left(\overset{(1A)}{\coth\left(\frac{\beta\omega_k}{2}\right)} - \overset{(1B)}{\coth\left(\frac{\beta}{2}(-ip^0+\omega_{k+p})\right)}\right)$$

2: $$-\frac{1}{n+i\frac{\beta\omega_k}{2\pi}}\,\frac{1}{n+i\frac{\beta}{2\pi}(-ip^0-\omega_{k+p})} \xrightarrow{\text{sum}} \frac{-2\pi^2}{\beta(\omega_k+ip^0+\omega_{k+p})}\left(\overset{(2A)}{\coth\left(\frac{\beta\omega_k}{2}\right)} - \overset{(2B)}{\coth\left(\frac{\beta}{2}(-ip^0-\omega_{k+p})\right)}\right)$$

3: $$-\frac{1}{n-i\frac{\beta\omega_k}{2\pi}}\,\frac{1}{n+i\frac{\beta}{2\pi}(-ip^0+\omega_{k+p})} \xrightarrow{\text{sum}} \frac{-2\pi^2}{\beta(-\omega_k+ip^0-\omega_{k+p})}\left(\overset{(3A)}{\coth\left(\frac{-\beta\omega_k}{2}\right)} - \overset{(3B)}{\coth\left(\frac{\beta}{2}(-ip^0+\omega_{k+p})\right)}\right)$$

4: $$\frac{1}{n-i\frac{\beta\omega_k}{2\pi}}\,\frac{1}{n+i\frac{\beta}{2\pi}(-ip^0-\omega_{k+p})} \xrightarrow{\text{sum}} \frac{2\pi^2}{\beta(-\omega_k+ip^0+\omega_{k+p})}\left(\overset{(4A)}{\coth\left(\frac{-\beta\omega_k}{2}\right)} - \overset{(4B)}{\coth\left(\frac{\beta}{2}(-ip^0-\omega_{k+p})\right)}\right)$$

Use $\coth(-x) = -\coth(x)$ to combine terms:

$$(1A) + (2A) = \left[\frac{2\pi^2}{\beta(\omega_k+ip^0-\omega_{k+p})} - \frac{2\pi^2}{\beta(-\omega_k+ip^0+\omega_{k+p})}\right]\coth\left(\frac{\beta\omega_k}{2}\right)$$

$$= \frac{2\pi^2}{\beta}\,\frac{-2\omega_{k+p}}{\omega_{k+p}^2-(\omega_k-ip^0)^2}\coth\left(\frac{\beta\omega_k}{2}\right)$$

and

$$(3A) + (4A) = \left[(1A) + (2A)\right]_{p^0\to -p^0}$$

Also:

$$(2B) + (4B) = \left[-\frac{2\pi^2}{\beta(\omega_k+ip^0-\omega_{k+p})} + \frac{2\pi^2}{\beta(-\omega_k+ip^0-\omega_{k+p})}\right]\coth\left(\frac{\beta(-ip^0+\omega_{k+p})}{2}\right)$$

$$= \frac{2\pi^2}{\beta}\,\frac{-2\omega_k}{\omega_k^2-(\omega_{k+p}-ip^0)^2}\coth\left(\frac{\beta(-ip^0+\omega_{k+p})}{2}\right)$$

and

$$(1B) + (3B) = \left[(2B) + (4B)\right]_{p^0\to -p^0}$$

So,

Pull $\frac{-1}{2}$ to front

$$-\Sigma(p^0,\vec{p}) = \frac{-g^2}{2\beta}\left(\frac{\beta}{2\pi}\right)^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{4\omega_k \omega_{k+p}} \left[① + ② + ③ + ④ \right]$$

$$= \frac{g^2}{8} \int \frac{d^3k}{(2\pi)^3} \left[\frac{1}{\omega_k} \coth\left(\frac{\beta\omega_k}{2}\right) \frac{1}{\omega_{k+p}^2 - (\omega_k - ip^0)^2} + \frac{1}{\omega_{k+p}} \coth\left(\frac{\beta\omega_{k+p}}{2}\right) \frac{1}{\omega_k^2 - (\omega_{k+p} - ip^0)^2} + (p^0 \to -p^0) \right]$$

where $\omega_k^2 = \vec{k}^2 + m^2$, $\omega_{k+p}^2 = (\vec{k}+\vec{p})^2 + m^2$.

Can isolate temperature dependent part from the temperature independent part (which is in need of renormalization) by using

$$\coth\left(\frac{\beta\omega_k}{2}\right) = 1 + 2n_B(\omega_k), \qquad n_B(\omega_k) = \frac{1}{e^{\beta\omega_k} - 1}$$

$$-\Sigma(p^0,\vec{p}) = \begin{pmatrix}\text{zero}\\ \text{temp.}\end{pmatrix} + \frac{g^2}{\cancel{8}_{4}} \int \frac{d^3k}{(2\pi)^3} \left[\frac{\cancel{2}n_B(\omega_k)}{\omega_k} \frac{1}{\omega_{k+p}^2 - (\omega_k - ip^0)^2} + \frac{\cancel{2}n_B(\omega_{k+p})}{\omega_{k+p}} \frac{1}{\omega_k^2 - (\omega_{k+p} - ip^0)^2} + (p^0 \to -p^0) \right]$$

One can evaluate this numerically by changing integration variables to spherical polars, with the z-axis aligned parallel with $\vec{p}$.

Then $(\vec{k}+\vec{p})^2 = |\vec{k}|^2 + |\vec{p}|^2 + |\vec{k}||\vec{p}|\cos\theta$,

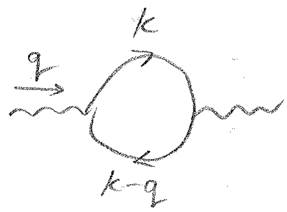
so that

$$\omega_{k+p} = \left[|\vec{k}|^2 + |\vec{p}|^2 + |\vec{k}||\vec{p}|\cos\theta + m^2 \right]^{1/2}$$

Then,

$$\int \frac{d^3k}{(2\pi)^3} \longrightarrow \frac{1}{(2\pi)^3} \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) \int d|\vec{k}|\, |\vec{k}|^2$$

$$= \frac{1}{(2\pi)^2} \int d|\vec{k}|\, |\vec{k}|^2 \int_{-1}^{1} d(\cos\theta)$$



$$\frac{\cancel{e^{-x/2}}(e^x-1)}{\cancel{e^{x/2}}(e^x+1)} = \frac{e^x}{e^x+1} - \frac{1}{e^x+1}$$

Write $\Pi_{\mu\nu} = D^{-1}_{\mu\nu} - D^{-1}_{F,\mu\nu}$ $\qquad k=$

$$\Pi^{\mu\nu} = e^2 \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \mathrm{Tr}\left[\gamma^\mu \not{k} \gamma^\nu (\not{k}-\not{q})\right] \frac{1}{k^2} \frac{1}{(k-q)^2}$$

$$\mathrm{Tr}[\ldots] = 4\left(k^\mu (k-q)^\nu - \delta^{\mu\nu} k\cdot(k-q) + (k-q)^\mu k^\nu\right)$$

$$= 4\left(k^\mu k^\nu - k^\mu q^\nu - \delta^{\mu\nu}(k^2 - k\cdot q) + k^\mu k^\nu - q^\mu k^\nu\right)$$

$$= 4\left(2k^\mu k^\nu - \delta^{\mu\nu} k^2 + O(q)\right)$$

$$\Pi^{\mu\nu} \sim e^2 \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3} \left(8k^\mu k^\nu - 4\delta^{\mu\nu} k^2\right) \frac{1}{k^2} \frac{1}{(k-q)^2}$$

$$= e^2 \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \left(\frac{8k^\mu k^\nu}{k^2 (k-q)^2} - \frac{4\delta^{\mu\nu}}{(k-q)^2}\right)$$

2nd term: $$= -4e^2 \delta^{\mu\nu} \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3} \underbrace{\frac{1}{k^2 + O(q)}}_{\frac{1}{\left(\frac{(2n+1)\pi}{\beta}\right)^2 + \vec{k}^2}}$$

Perform frequency sum:

$$= -4e^2 \delta^{\mu\nu} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2|\vec{k}|} \mathrm{Tanh}\left(\frac{|\vec{k}|\beta}{2}\right)$$

$$= -4e^2 \delta^{\mu\nu} \underbrace{\int \frac{d^3k}{(2\pi)^3} \frac{1}{2|\vec{k}|} \left(1 - 2n_F(|\vec{k}|)\right)}_{= -\frac{T^2}{24} + \text{UV divergence.}}$$

$$= +\frac{e^2 T^2}{6} \delta^{\mu\nu}$$

Matsubara representation

BOSONIC

$$\coth\left(\frac{\beta\omega_k}{2}\right) = 1 + 2n_B(\omega_k) = \beta\omega_k \sum_{n=-\infty}^{+\infty} \frac{2}{(2\pi)^2 n^2 + \beta^2\omega_k^2}$$

FERMIONIC

$$\tanh\left(\frac{\beta\omega_k}{2}\right) = 1 - 2n_F(\omega_k) = \beta\omega_k \sum_{n=-\infty}^{+\infty} \frac{2}{(2\pi)^2 (n+\frac{1}{2})^2 + \beta^2\omega_k^2}$$

Property:

$$n_F(\omega_k) = \frac{1}{e^{\beta\omega_k}+1} = 1 - n_F(-\omega_k)$$

$$n_B(\omega_k) = \frac{1}{e^{\beta\omega_k}-1} = -(1 + n_B(-\omega_k))$$