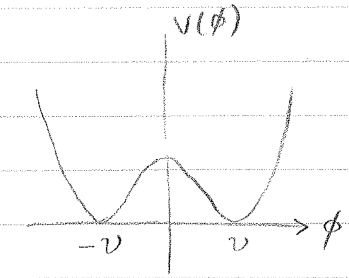
Topological Solitons – Kink (1+1)-D

Theory: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4}\phi^4$



$\phi \equiv$ dimensionless

$[\mu] = [\text{Energy}]$

$[\lambda] = [\text{Energy}]^2$

Classical vacua at:

$$\langle\phi\rangle = \pm v = \pm\frac{\mu}{\sqrt{\lambda}}$$

Perterbations around v: $\quad \phi \to v + \phi'$

$$V(\phi) = -\frac{1}{2}\mu^2(v+\phi')^2 + \frac{\lambda}{4}(v+\phi')^4$$

$$= \frac{1}{2}(-\mu^2 + 3\lambda v^2)\phi'^2 + \dots$$

$$= \frac{1}{2}(2\mu^2)\phi'^2 + \dots \quad \longleftarrow \text{quanta of mass} = \sqrt{2}\,\mu.$$

Find static solution of field equations interpolating between $\phi = +v$ & $\phi = -v$, with finite energy:

$$H = \int d^3x\left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4\right)$$

$$H_{stat} = \int d^3x\left(\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4\right) = E[\phi] \longleftarrow \text{"Energy Functional"}$$

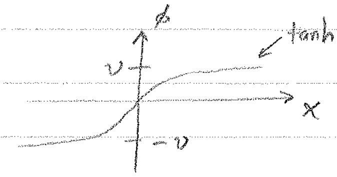
Minimize functional:

$$\frac{\delta E}{\delta\phi} = \frac{\partial E}{\partial\phi} - \frac{d}{dx}\left(\frac{\partial E}{\partial\left(\frac{d\phi}{dx}\right)}\right) = 0$$

$$= -\mu^2\phi + \lambda\phi^3 - \frac{d}{dx}\left(\frac{d\phi}{dx}\right) = 0$$

or

$$\boxed{\frac{d^2\phi}{dx^2} = \mu^2\phi - \lambda\phi^3}$$



Try (ansatz):

$$\phi_K(x) = v \tanh\left(\gamma(x - x_0)\right)$$

Solve for γ — center of "Kink"

Differential equation becomes:

$$\frac{d^2}{dx^2} v \tanh(\gamma(x-x_0)) = \mu^2 v \tanh(\gamma(x-x_0)) - \lambda v^3 \tanh^3(\gamma(x-x_0))$$

Using $v = \mu/\sqrt{\lambda}$ in RHS,

$$-2v\gamma^2 \operatorname{sech}^2(\gamma(x-x_0)) \tanh(\gamma(x-x_0))$$
$$= -v^3 \lambda \operatorname{sech}^2(\gamma(x-x_0)) \tanh(\gamma(x-x_0))$$

$$-2v\gamma^2 = -v^3\lambda \quad \Rightarrow \quad \gamma = \sqrt{\frac{v^2\lambda}{2}} = v\sqrt{\frac{\lambda}{2}}$$

So, kink solution is

$$\phi_K(x) = v \tanh\left(v\sqrt{\frac{\lambda}{2}}(x - x_0)\right)$$