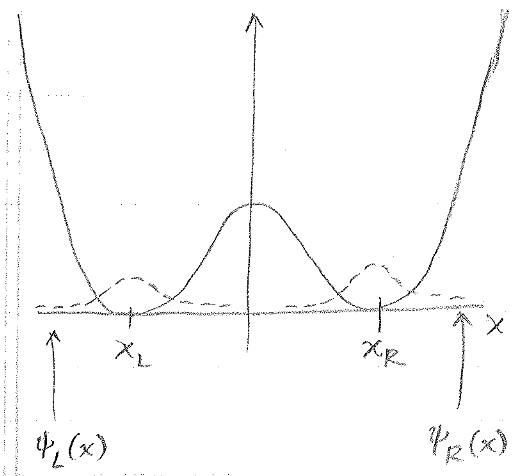
Tunneling in Potentials with Classical Degeneracy

Consider:

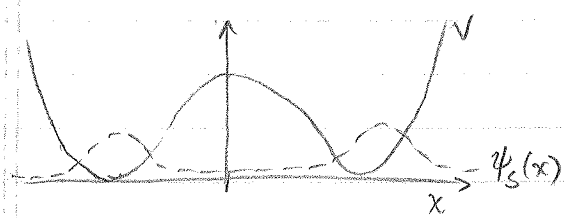


$$V(x) = -\frac{\mu^2}{2}x^2 + \frac{\lambda}{4}x^4$$

Has two degenerate minima: x_L & x_R.
Looking at just one-half of the potential, the G.S would be $\psi_L(x)$ & $\psi_R(x)$.

However, there is a symmetry P: $x \leftrightarrow -x$, generator $\hat{P}$:

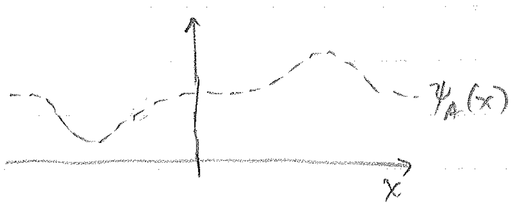
$\hat{P}\psi(x) = \psi(-x)$ e. values ± 1. and $[\hat{H}, \hat{P}]$ (commutes with Hamil.)



E-states of $\hat{H}$ are also e-states of $\hat{P}$

$$\psi_S(x) = \frac{1}{\sqrt{2}}\left(\psi_R(x) + \psi_L(x)\right)$$

$$\psi_A(x) = \frac{1}{\sqrt{2}}\left(\psi_R(x) - \psi_L(x)\right)$$



These can't be degenerate: From general arguments of Quantum Mechanics, the G.S. wave function can always be taken to be positive definite.

Tunneling between the two breaks the symmetry:
What is the difference $= E_A - E_S = ?$

Calculate energy splitting. Start with time-indep. Schrödinger eqn.

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_A + V(x)\psi_A = E_A\psi_A\right)\times\psi_S$$
$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_S + V(x)\psi_S = E_S\psi_S\right)\times\psi_A$$

multiply by ψ_A & ψ_S.

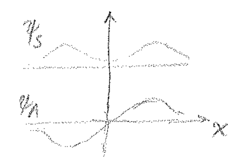
Subtract: $\binom{\text{first}}{\text{eqn}} - \binom{\text{second}}{\text{eqn}}$

$$-\frac{\hbar^2}{2m}\left(\psi_S\frac{d^2\psi_A}{dx^2} - \psi_A\frac{d^2\psi_S}{dx^2}\right) = (E_A - E_S)\psi_S\psi_A$$

cross terms cancel

$$-\frac{\hbar^2}{2m}\frac{d}{dx}\left(\psi_S\frac{d\psi_A}{dx} - \psi_A\frac{d\psi_S}{dx}\right) =$$

pull out a total derivative.



Now, integrate over x from $-\infty$ to 0 (region where ψ_S & ψ_A destructively interfere

$$-\frac{\hbar^2}{2m}\underbrace{\int_{-\infty}^{0}\frac{d}{dx}\left(\psi_S\frac{d\psi_A}{dx} - \psi_A\frac{d\psi_S}{dx}\right)dx}_{\text{use fund. th. of Calc. } \psi_S\psi_A' - \psi_A\psi_S' \text{ vanishes at } -\infty.} = (E_A - E_S)\underbrace{\int_{-\infty}^{0} dx\,\psi_S\psi_A}_{-\frac{1}{2}}$$

dest. interference

half-normalization

$$-\frac{\hbar^2}{2m}\left(\psi_S\frac{d\psi_A}{dx} - \psi_A\frac{d\psi_S}{dx}\right)_{x=0} = (E_A - E_S)\left(-\frac{1}{2}\right)$$

Look at sketch: $\psi_S'(0) = 0$ $\qquad \psi_S(0) = \frac{1}{\sqrt{2}}\left(\psi_R(0) + \psi_L(0)\right) = \sqrt{2}\,\psi_L(0)$ (equal)

$\psi_A(0) = 0$ (node) $\qquad \psi_A'(0) = \sqrt{2}\,\psi_L'(0)$.

$$(-2)\times\frac{-\hbar^2}{2m}(\sqrt{2})^2\psi_L(0)\psi_L'(0) = E_A - E_S.$$

$$+\frac{2\hbar^2}{m}\underbrace{\psi_L(0)\psi_L'(0)}_{\text{Use WKB to estimate this}} = E_A - E_S.$$

(use sol'n in cl. forbidden reg.)

WKB: $\psi_L(x) = \dfrac{C}{|2mV(x)|^{1/4}}\exp\left[-\frac{1}{\hbar}\int_{x_L}^{x} dx'\sqrt{2mV(x')}\right]$

$\psi_L'(x) \sim \exp\left[-\frac{1}{\hbar}\int_{x_L}^{x} dx'\sqrt{2mV(x')}\right]$

So,

$$\psi_L(0)\,\psi_L'(0) \sim \psi_L^2(0) \sim \exp\left[\frac{-2}{\hbar}\int_{x_L}^{0} dx\,\sqrt{2mV(x)}\right]$$

Therefore, $\boxed{E_A - E_S \sim \frac{2\hbar^2}{m}\exp\left[\frac{-2}{\hbar}\int_{x_L}^{0} dx\,\sqrt{2mV(x)}\right]}$ is the splitting.

A more symmetric form can be obtained (needed in generalization to multivariables):

Use $\psi_S(0) = \frac{1}{\sqrt{2}}\left(\psi_R(0) + \psi_L(0)\right)$ and $\psi_A'(0) = \frac{1}{\sqrt{2}}\left(\psi_R'(0) - \psi_L'(0)\right)$

Then
$$\left(\psi_S \frac{d\psi_A}{dx}\right)_{x=0} = \frac{1}{2}\left(\psi_R(0) + \psi_L(0)\right)\left(\psi_R'(0) - \psi_L'(0)\right)$$
$$= \frac{1}{2}\left(\underbrace{\psi_R\psi_R' - \psi_L\psi_L'}\; - \psi_R\psi_L' + \psi_L\psi_R'\right)_{x=0}.$$

since $\psi_R'(0) = -\psi_L'(0)$ write: $-\psi_R\psi_L' + \psi_L\psi_R'$

$$= \frac{1}{2}\left(2\times\left(\psi_L\psi_R' - \psi_R\psi_L'\right)\right)_{x=0} = \psi_L\frac{d\psi_R}{dx} - \psi_R\frac{d\psi_L}{dx}$$

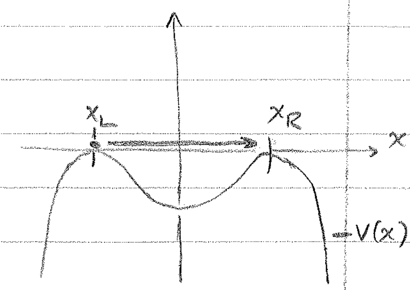
Then,
$$E_A - E_S = \left(-\frac{1}{2}\right)\times\frac{-\hbar^2}{2m}\left(\psi_L\frac{d\psi_R}{dx} - \psi_R\frac{d\psi_L}{dx}\right)_{x=0}$$
$$= \frac{\hbar^2}{m}\left(\psi_L\frac{d\psi_R}{dx} - \psi_R\frac{d\psi_L}{dx}\right)_{x=0},$$
which can be estimated using WKB.

Therefore, in systems with multiple classical degenarate minima, "quantum fluctuations" will break this degeneracy to give a set of states: the one whose wavefunction is positive definite everywhere is the ground state.

Tunneling in Potentials with Classical Degeneracy (Euclidean action approach)

Start with $S[x]$, and construct the Euclidean action

$$S_E = \int d\tau \left[\frac{1}{2} m \left(\frac{\partial x}{\partial \tau}\right)^2 + V(x) \right]$$



Extremize the action: $m \dfrac{d^2x}{d\tau^2} = \dfrac{dV}{d\tau}$,

but this time there is no bounce solution.

Instead find solution that goes from $x_L(-\infty) \rightarrow x_R(+\infty)$,
subject to $x_L'(-\infty) = 0$ and $x_R'(+\infty) = 0$ (by cons. of Eucl. energy).

This solution is the instanton $x_I(\tau)$ (does not return to x_L).

Semiclassical tunneling exponential:

$$T \sim e^{-S_E[x_I]/\hbar} = \exp\left(\frac{-1}{\hbar} \int_{-\infty}^{\infty} d\tau \left[\frac{1}{2} m \left(\frac{dx}{d\tau}\right)^2 + V(x) \right] \right)$$

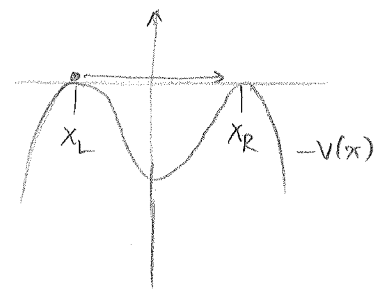
and by symmetry, $\text{action}(x_L \rightarrow 0) = \text{action}(x_R \rightarrow 0)$

$$= \exp\left(\frac{-2}{\hbar} \int_{-\infty}^{0} d\tau \left[\frac{1}{2} m \left(\frac{dx}{d\tau}\right)^2 + V(x) \right] \right)$$

Tunneling in Potentials with Classical Degeneracy (Euclidean Action approach)

Start with action $S[\vec{x}]$, and construct the Euclideanized Action

$$S_E = \int d\tau \left[\frac{1}{2} m \left(\frac{d\vec{x}}{d\tau}\right)^2 + V(\vec{x}) \right]$$



Extremize the action: $\quad m \dfrac{d^2\vec{x}}{d\tau^2} = \dfrac{dV}{d\vec{x}}$,

but this time, there is no bounce solution.

Instead find solution that goes from $x_L(-\infty) \to x_R(+\infty)$,
subject to $x_L'(-\infty) = 0$ and $x_R'(+\infty) = 0$ (by cons. of eucl. energy)

This solution is the Instanton, $x_I(\tau)$. (does not return to x_L).

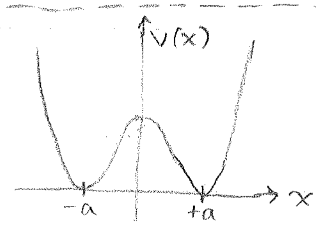
Semiclassical tunneling exponential:

$$T \sim e^{-S_E[x_I]/\hbar} = \exp\left[\frac{-1}{\hbar} \int_{-\infty}^{+\infty} d\tau \left[\frac{1}{2} m \left(\frac{d\vec{x}_I}{d\tau}\right)^2 + V(\vec{x}_I) \right] \right].$$

and by symmetry, action $(x_L \to 0)$ = action $(0 \to x_R)$.

$$= \exp\left[-\frac{2}{\hbar} \int_{-\infty}^{0} d\tau \left[\frac{1}{2} m \left(\frac{d\vec{x}_I}{d\tau}\right)^2 + V(\vec{x}_I) \right] \right]$$

Explicit calculation for $V(x) = \frac{g}{4}(x^2 - a^2)^2$



Note: Harmonic frequency at bottom of well:

$$m\omega_0^2 = \frac{d^2}{dx^2} V(x=a) = 2g a^2$$

$$\Rightarrow \omega_0 = \sqrt{\frac{8a^2\lambda}{m}} \qquad \Rightarrow \quad a^2 = \frac{m\omega_0^2}{2g}$$

Maybe better to write: $V(x) = \frac{g}{4}\left(x^2 - \frac{m\omega_0^2}{2g}\right)$

Search for solutions to $m \dfrac{d^2x}{d\tau^2} = g\,(x^2 - a^2)\,x$,

subject to boundary condition: $x(-\infty) = -a$, $\quad x'(-\infty) = 0$.

<u>Try</u>: $x_I(t) = a \tanh(\gamma(\tau - \tau_0))$

Solve for γ

center of instanton is arbitrary — time-translation invariance

$\tau_0 \equiv$ <u>instanton displacement</u>

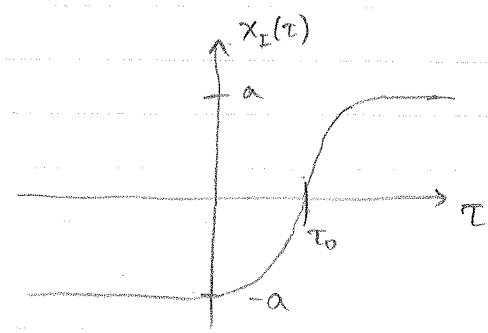
Differential equation becomes:

$$m \frac{d^2}{d\tau^2}\Big(a \tanh(\gamma(\tau-\tau_0))\Big) = g\left(a^2 \tanh^2(\gamma(\tau-\tau_0)) - a^2\right) \times a \tanh(\gamma(\tau-\tau_0))$$

$$-2 m a \gamma^2\, \cancel{\operatorname{sech}^2(\gamma(\tau-\tau_0))}\, \tanh(\gamma(\tau-\tau_0)) = -g a^3\, \cancel{\operatorname{sech}^2(\gamma(\tau-\tau_0))}\, \tanh(\gamma(\tau-\tau_0))$$

$$\Rightarrow \quad \gamma = \sqrt{\frac{g a^2}{2m}} = \frac{\omega_0}{2}$$

Hence $\quad \underline{x_I(\tau) = a \tanh\left(\frac{\omega_0}{2}(\tau - \tau_0)\right)}$



Hence the semiclassical exponential factor is:

$$\Gamma \sim e^{-S_E[x_I]/\hbar} = \exp\left[-\frac{2}{\hbar}\int_{-\infty}^{\tau_0} d\tau \left[\frac{1}{2} m \left(\frac{dx_I}{d\tau}\right)^2 + \frac{g}{4}(x_I^2 - a^2)^2\right]\right]$$

$$= \exp\left[-\frac{2}{\hbar}\int_{-\infty}^{\tau_0} d\tau \; \frac{m^2\omega_0^4}{8g} \operatorname{sech}^4\left(\tfrac{1}{2}\omega_0(\tau-\tau_0)\right)\right]$$

$$= \exp\left[-\frac{m^2\omega_0^3}{3g\hbar}\right], \qquad \omega_0^2 = \frac{8a^2\lambda}{m}$$ is the fundamental frequency at the bottom of well.