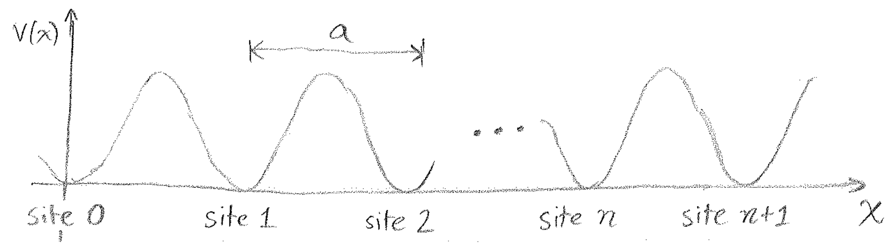
Periodic Potentials



Hamiltonian:

$$\hat{H} = \frac{p^2}{2m} + V(x)$$

Property: $V(x+a) = V(x)$

$\Rightarrow$ Hamiltonian, $\hat{H}$, has the property $\underbrace{\hat{t}(a)}_{\text{translation operator}} \hat{H}\, t^{\dagger}(a) = \hat{H} \quad \Rightarrow \quad [\hat{H}, \hat{t}(a)] = 0.$

In the limit of infinite barrier height, a state, $|n\rangle$, localized inside one well at site n would a stationary state:

$$\hat{H}\,|n\rangle = E_{GS}\,|n\rangle \qquad \text{note: } \langle n|n+1\rangle = 0 \quad (\text{No overlap})$$

The state is a ground state of the system (infinitely degenerate). But $|n\rangle$ is not an e-state of $\hat{t}(a)$:

$$\hat{t}(a)\,|n\rangle = |n+1\rangle.$$

Since $\hat{t}(a)$ and $\hat{H}$ commute, we should be able to find states that are simultaneous e-states of $\hat{H}$ and $\hat{t}(a)$. Consider the superposition

$$|\theta\rangle = \sum_{n=-\infty}^{+\infty} e^{-in\theta}\,|n\rangle, \qquad -\pi \le \theta \le \pi$$

(in the extreme limit of $|\theta| = \pi$, adjacent states nearly alternate phase)

Then $|\theta\rangle$ is e-state of $\hat{t}(a)$

$$\hat{t}(a)\,|\theta\rangle = \sum_{n=-\infty}^{+\infty} e^{-in\theta}\,|n+1\rangle = \sum_{n=-\infty}^{\infty} e^{-i(n-1)\theta}\,|n+1\rangle = \underbrace{e^{i\theta}}_{\text{eigenvalue}}\,|\theta\rangle,$$

ch. variables: $n \to n-1$

So the set of states labeled by $|\theta\rangle$ are simultaneous e-states of the Hamiltonian, $\hat{H}$, and the translation operator, $\hat{t}(a)$.