Euclideanization of Maxwell Action

$$iS[A] = i\int d^4x \; -\frac{1}{4} F_{\mu\nu}F^{\mu\nu}$$

$$= i\int d^4x \; -\frac{1}{2}\,\partial_\mu A_\nu\,(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= i\int d^4x \; -\frac{1}{2}\Big[\partial_0 A_i\,(\partial^0 A^i - \partial^i A^0) + \partial_i A_0\,(\partial^i A^0 - \partial^0 A^i) + \partial_i A_j\,(\partial^i A^j - \partial^j A^i)\Big]$$

lower indices.

$$= i\int d^4x \; -\frac{1}{2}\Big[-\partial_0 A_i\,(\partial_0 A_i - \partial_i A_0) - \partial_i A_0\,(\partial_i A_0 - \partial_0 A_i) + \partial_i A_j\,(\partial_i A_j - \partial_j A_i)\Big]$$

Euclideanize:

Take $t = -i\tau$

and $A_0 = i\hat{A}_0 \qquad A_i = \hat{A}_i$

(so that field eqns do not become complex)

$$\partial_0 = \frac{\partial}{\partial t} = \frac{\partial}{\partial(-i\tau)} = i\,\partial_\tau$$

$$\vec{E} = -\vec{\nabla}A^0 - \frac{\partial\vec{A}}{\partial t}$$

$$= -i\vec{\nabla}\hat{A}^0 - i\frac{\partial\vec{A}}{\partial\tau} = i\hat{E}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \hat{B}$$

$$iS[A] \longrightarrow i\int d(-i\tau)\,d^3x \; -\frac{1}{2}\Big[-i\,\partial_\tau\hat{A}_i\,(i\partial_\tau\hat{A}_i - \partial_i(i\hat{A}_0)) - \partial_i(i\hat{A}_0)\big(\partial_i(i\hat{A}_0) - i\partial_0 A_i\big)$$

$$+ \partial_i A_j\,(\partial_i A_j - \partial_j A_i)\Big]$$

$$= -\int d\tau\, d^4x \; +\frac{1}{2}\Big[\partial_\tau\hat{A}_i(\partial_\tau\hat{A}_i - \underbrace{\partial_i\hat{A}_0}) + \partial_i\hat{A}_0(\partial_i\hat{A}_0 - \underbrace{\partial_\tau\hat{A}_i}) + \partial_i\hat{A}_j\,(\partial_i\hat{A}_j - \partial_j\hat{A}_i)\Big]$$

exponential damping.

single τ-derivative (removed in axial gauge).

$$= -\underbrace{\int (d^4x)_E \; \overbrace{\frac{1}{4}(\hat{F}_{\mu\nu}\hat{F}_{\mu\nu})}^{}}_{S_E[A]} = -S_E[A]$$

$$\frac{1}{4}(\hat{F}_{\mu\nu}\hat{F}_{\mu\nu}) = \mathcal{L}_E$$

Axial gauge: $\hat{A}_{\mu=0} = 0$

$$\hat{F}_{ij} = \partial_i\hat{A}_j - \partial_j\hat{A}_i$$

$$iS[A] = -S_E[A] = -\int (d^4x)_E\left[\frac{1}{2}\left(\frac{\partial\hat{A}_i}{\partial\tau}\right)^2 + \frac{1}{2}\hat{F}_{ij}\hat{F}_{ij}\right].$$

Commutator of covariant derivatives

$$[D_\mu, D_\nu] = [\partial_\mu + ieA_\mu, \partial_\nu + ieA_\nu]$$

$$= \partial_\mu(ieA_\nu \,\cdot) - ieA_\nu \partial_\mu + ieA_\mu \partial_\nu - \partial_\nu(ieA_\mu \,\cdot)$$

$$= ie\,\partial_\mu A_\nu + \cancel{ieA_\nu \partial_\mu} - \cancel{ieA_\nu \partial_\mu}$$

$$\cancel{+ ieA_\mu \partial_\nu} - ie\,\partial_\nu A_\mu - \cancel{ieA_\mu \partial_\nu}$$

$$= ie\underbrace{(\partial_\mu A_\nu - \partial_\nu A_\mu)}_{(F_{\mu\nu})_E}$$