Euclideanization of Non-abelian Yang-Mills Action

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}, \qquad F_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\mu A^a_\nu - g f^{abc} A^b_\mu A^c_\nu$$

$$= -\frac{1}{4} \overset{①}{(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a})}$$

$$+ \overset{②}{g} f^{abc} (\partial_\mu A^a_\nu) A^{\mu b} A^{\nu c} - \overset{③}{\frac{g^2}{4}} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e}$$

① First term Euclideanizes same way as Maxwell action.

② $$g f^{abc} (\partial_\mu A^a_\nu) A^{\mu b} A^{\nu c} = g f^{abc} \Big[(\partial_0 A^a_0) A^{0b} A^{0c} + (\partial_i A^a_0) A^{ib} A^{0c} + (\partial_0 A^a_i) A^{0b} A^{ic} + (\partial_i A^a_j) A^{ib} A^{jc} \Big]$$

$$= g f^{abc} \Big[(\partial_0 A^a_0) A^b_0 A^c_0 - (\partial_i A^a_0) A^b_i A^c_0 - (\partial_0 A^a_i) A^b_0 A^c_i + (\partial_i A^a_j) A^b_i A^c_j \Big]$$

Euclideanize: $t \to -i\tau \Rightarrow \partial_0 \to i\partial_\tau$ & $A_0 \to i\hat{A}_0$

② $$\longrightarrow \underset{\substack{\text{from } d(-i\tau) \\ \text{in action}}}{-g} f^{abc} \Big[(i\partial_\tau)(i\hat{A}_0)(i\hat{A}^b_0)(i\hat{A}^c_0) - (\partial_i\, i\hat{A}^a_0) \hat{A}^b_i (i\hat{A}^c_0) - (i\partial_\tau \hat{A}^a_i)(i\hat{A}^b_0) \hat{A}^c_i + (\partial_i \hat{A}^a_j) \hat{A}^b_i \hat{A}^c_j \Big]$$

$$= -g f^{abc} \Big[(\partial_\tau \hat{A}^a_0) \hat{A}^b_0 \hat{A}^c_0 + (\partial_i \hat{A}^a_0) \hat{A}^b_i \hat{A}^c_0 + (\partial_\tau \hat{A}^a_i) \hat{A}^b_0 \hat{A}^c_i + (\partial_i \hat{A}^a_j) \hat{A}^b_i \hat{A}^c_j \Big]$$

additional single time derivatives

$$\equiv -g f^{abc} \Big[(\partial_\mu \hat{A}^a_\nu) \hat{A}^b_\mu \hat{A}^c_\nu \Big]_E$$

Similarly,

③ $$\longrightarrow \underset{\substack{\text{from } d(-i\tau) \\ \text{in action}}}{(-1)} \times -\frac{g^2}{4} f^{abc} f^{ade} \Big[\hat{A}^b_0 \hat{A}^c_0 \hat{A}^d_0 \hat{A}^e_0 + \hat{A}^b_0 \hat{A}^c_i \hat{A}^d_0 \hat{A}^e_i + \hat{A}^b_i \hat{A}^c_0 \hat{A}^d_i \hat{A}^e_0 + \hat{A}^b_i \hat{A}^c_j \hat{A}^d_i \hat{A}^e_j \Big]$$

$$= +\frac{g^2}{4} f^{abc} f^{ade} \Big[\hat{A}^b_\mu \hat{A}^c_\nu \hat{A}^d_\mu \hat{A}^e_\nu \Big]_E$$

So,

$$\mathcal{L}_E = +\frac{1}{4} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) - g f^{abc} \Big[(\partial_\mu A^a_\nu) A^b_\mu A^c_\nu \Big]_E$$

$$+ \frac{g^2}{4} f^{abc} f^{ade} \Big[A^b_\mu A^c_\nu A^d_\mu A^e_\nu \Big]_E \equiv \boxed{+\frac{1}{4} (F^a_{\mu\nu} \cdot F^a_{\mu\nu})_E}$$

Euclideanization of Topological term

$$iS_\theta[A] = i\int d^4x \, \frac{-g^2\theta}{32\pi^2} F^a_{\mu\nu}\tilde{F}^{\mu\nu\, a}$$

$$= i\int d^4x \, \frac{-g^2\theta}{32\pi^2}\left\{\epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(A^a_\nu F^a_{\rho\sigma} + \frac{1}{3} g f^{abc} A^a_\nu A^b_\rho A^c_\sigma\right)\right\}$$

lower indices.

$$= i\int d^4x \, \frac{+g^2\theta}{32\pi^2}\left\{\cancel{+}\,\epsilon_{\mu\nu\rho\sigma}\partial_\mu\left(A^a_\nu F^a_{\rho\sigma} + \frac{1}{3} g f^{abc} A^a_\nu A^b_\rho A^c_\sigma\right)\right\}$$

Euclideanize:

Define $\hat{\epsilon}_{\mu\nu\rho\sigma} = \epsilon_{\mu\nu\rho\sigma} = \begin{cases} -1 & \text{even perm } 0,1,2,3 \\ +1 & \text{odd perm } 0,1,2,3 \end{cases}$ (matches lit. $\epsilon_{1234} = +1$)

Also, $t \to -i\tau$

$A_0 = i\hat{A}_0$

Because of $\epsilon_{\mu\nu\rho\sigma}$, only one of the factors of $\partial_\mu A_\nu F_{\rho\sigma}$ will be 0-component

$\Rightarrow$ only one factor of i.

$$iS_\theta[A] = i\int d(-i\tau)\, d^3x \, \frac{ig^2\theta}{32\pi^2}\left\{\hat{\epsilon}_{\mu\nu\rho\sigma}\hat{\partial}_\mu\left(\hat{A}^a_\nu \hat{F}^a_{\rho\sigma} + \frac{1}{3} g f^{abc}\hat{A}^a_\nu \hat{A}^b_\rho \hat{A}^c_\sigma\right)\right\}.$$

$$= -\int d\tau\, d^3x \, \frac{-ig^2\theta}{32\pi^2}\,\hat{\epsilon}_{\mu\nu\rho\sigma}\hat{\partial}_\mu\left[\hat{A}^a_\nu \hat{F}^a_{\rho\sigma} + \frac{1}{3} g f^{abc}\hat{A}^a_\nu \hat{A}^b_\rho \hat{A}^c_\sigma\right]$$

$$= -\int (d^4x)_E \; -i\theta\frac{g^2}{32\pi^2}\hat{F}^a_{\mu\nu}\hat{\tilde{F}}^{\mu\nu\, a}$$

or

$$= -\int (d^4x)_E \; -i\theta\frac{g^2}{16\pi^2}\mathrm{Tr}\left[\hat{F}_{\mu\nu}\hat{\tilde{F}}^{\mu\nu}\right]$$

Summary of Euclideanized Yang-Mills

$$\left.\begin{array}{l} \vec{E} \longrightarrow i\hat{E} \\ \vec{B} \longrightarrow \hat{B} \end{array}\right\} \quad \begin{array}{l} F_{\mu\nu}F^{\mu\nu} = -2(\vec{E}^2 - \vec{B}^2) = +2(\hat{E}^2 + \hat{B}^2) \\ F_{\mu\nu}\tilde{F}^{\mu\nu} = -4\vec{E}\cdot\vec{B} = -4i\,\hat{E}\cdot\hat{B} \end{array}$$

Then

$$\mathcal{L}_{YM} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \frac{g^2\theta}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{\mu\nu a}$$

$$\mathcal{L} = -\mathcal{L}_E$$

$$iS = -S_E = \int (d^4x)_E\, \mathcal{L}_E$$

$$\mathcal{L}_E = +\frac{1}{2}\left((\vec{E}^a)^2 + (\vec{B}^a)^2\right) - i\theta\frac{g^2}{32\pi^2}\hat{F}^a_{\mu\nu}\hat{\tilde{F}}^{\mu\nu a}$$

Euclideanized field strengths:

$$\hat{F}_{\mu\nu} = \begin{pmatrix} 0 & i\hat{E}_x & i\hat{E}_y & i\hat{E}_z \\ & 0 & -\hat{B}_z & \hat{B}_y \\ & & 0 & -\hat{B}_x \\ & & & 0 \end{pmatrix} \qquad \hat{\tilde{F}}_{\mu\nu} = \begin{pmatrix} 0 & \hat{B}_x & \hat{B}_y & \hat{B}_z \\ & 0 & i\hat{E}_z & -i\hat{E}_y \\ & & 0 & i\hat{E}_x \\ & & & 0 \end{pmatrix}$$