Instantons in SU(N) (pure) Yang-Mills Theory

Would like to calculate the overlap between equivalent vacua of winding numbers differing by 1: $\Delta n = 1$

$$\langle n+1 | \hat{H} | n \rangle$$

We look for the instanton solution to the Euclideanized Yang-Mills action:

$$S_E[A] = \int d^4x_E \ \frac{+1}{4}\left(F^a_{\mu\nu} F^a_{\mu\nu}\right)_E = \int d^4x_E \ \frac{+1}{2}\left((\vec{E}^a_E)^2 + (\vec{B}^a_E)^2\right)$$

(strictly positive)

That is, we aim to extremize the action subject to the boundary condition:

System starts at n:

$$A^{inst}_\mu(\tau \to -\infty) = \frac{-i}{g} U^{(n)}(\vec{x}) \, \partial_\mu U^{\dagger(n)}(\vec{x})$$

$\uparrow$ Eucl. time

Pure gauges

System ends at $n+1$:

$$A^{inst}_\mu(\tau \to +\infty) = \frac{-i}{g} U^{(n+1)}(\vec{x}) \, \partial_\mu U^{\dagger(n+1)}(\vec{x})$$

AND

$$A^{inst}_\mu(\vec{x} \to \infty) = 0 \quad \Rightarrow \quad \partial_\mu U(\hat{x}) = 0$$

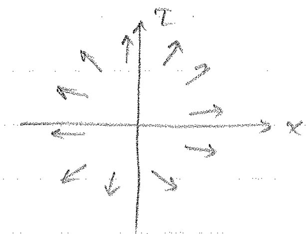
i.e. the $U(\hat{x})$ approaches a constant as $\vec{x} \to \infty$.

These boundary conditions can more easily be satisfied if we consider (hyper) spherically symmetric field configurations, such that gauge fields on the boundary 3-sphere have winding number $= 1$.

As with other large field configuration (topological) solutions, set angles on the hypersphere equal to angles on the vacuum manifold:

Let $U(x_\mu) = a_0(x_\mu)\,\mathbb{1}_{2\times 2} + i\vec{a}(x_\mu).\vec{\sigma}$

Subject to $\sum_{\mu=0}^{3} a_\mu a_\mu = 1$



Take $a_\mu = \hat{x}_\mu$, $\longleftarrow$ unit 4-vector

so that $U(\hat{x}_\mu) = \sum_\mu \hat{x}_\mu \sigma_\mu$, $\quad \sigma_\mu = (1, i\vec{\sigma}) \longleftarrow$ obey: $\sigma_\mu^\dagger \sigma_\nu + \sigma_\nu^\dagger \sigma_\mu = 2\delta_{\mu\nu}$

Hence, the gauge fields on the hypersphere at infinity is:

$$A_\mu(|x_\mu| \to \infty) = \frac{-i}{g} U(\hat{x})\, \partial_\mu U^\dagger(\hat{x})$$

$$= \frac{-i}{g} \hat{x}_\nu \sigma_\nu\, \partial_\mu(\hat{x}_\rho \sigma_\rho^\dagger)$$

$\Big\downarrow$ use $\partial_\mu \hat{x}_\rho = \dfrac{\delta_{\mu\rho} - \hat{x}_\mu \hat{x}_\rho}{|x|}$

$$= \frac{-i}{g} \hat{x}_\nu \sigma_\nu \left(\frac{\delta_{\mu\rho} - \hat{x}_\mu \hat{x}_\rho}{|x|}\right) \sigma_\rho^\dagger$$

$$= \frac{-i}{g} \underbrace{\sigma_\nu \sigma_\rho^\dagger}\, \hat{x}_\nu \left(\frac{\delta_{\mu\rho} - \hat{x}_\mu \hat{x}_\rho}{|x|}\right)$$

Now: $\sigma_\mu \sigma_\nu^\dagger = \left(\begin{array}{c|c} 1 = \delta_{00}\mathbb{1} & -i\vec{\sigma}_j \\ \hline i\vec{\sigma}_i & \sigma_i \sigma_j \end{array}\right)$

Note: $\sigma_\nu^\dagger = (1, -i\vec{\sigma})$, $\vec{\sigma}$ hermitian

write: $\sigma_i \sigma_j = \frac{1}{2}\{\sigma_i, \sigma_j\} + \frac{1}{2}[\sigma_i, \sigma_j]$

$$= \frac{1}{2}\, 2\delta_{ij} + \frac{1}{2}\, 2i\epsilon^a{}_{ij}\sigma^a$$

$$= \delta_{ij} + i\epsilon^a{}_{ij}\sigma^a$$

$a = \{1,2,3\}$
$\mu,\nu = \{0,1,2,3\}$

so, $\sigma_\mu \sigma_\nu^\dagger = \left(\begin{array}{c|c} \delta_{00} & 0 \\ \hline 0 & \delta_{ij} \end{array}\right) + i \left(\begin{array}{c|c} 0 & -\delta_{\mu 0}\delta^a{}_\nu \\ \hline \delta_{\nu 0}\delta^a{}_\mu & \epsilon^a{}_{ij} \end{array}\right)\sigma^a$

$$= \delta_{\mu\nu} + i\underbrace{\left(\epsilon^a{}_{\mu\nu} + \delta^a{}_\mu \delta_{\nu 0} - \delta^a{}_\nu \delta_{\mu 0}\right)}\sigma^a = \delta_{\mu\nu} + i\eta^a{}_{\mu\nu}\sigma^a$$

$(\epsilon^a = 0$ if $\{\mu = 0\})$ $\quad = \eta^a$ 't Hooft symbols 't Hooft symbols   Maps $\underbrace{SO(3)}_{\eta^a_{\mu\nu}} \times \underbrace{SO(3)}_{\bar{\eta}^a_{\mu\nu}} \longrightarrow SO(4)$

Def$^{\underline{n}}$ $\boxed{\eta^a{}_{\mu\nu} = \epsilon^a{}_{\mu\nu} + \delta^a{}_\mu \delta_{\nu 0} - \delta^a{}_\nu \delta_{\mu 0}}$

There are three $(a = \{1,2,3\})$ 4x4 matrices $(\mu,\nu = \{0,1,2,3\})$

$\epsilon^a{}_{\mu\nu} = 0$ if $\{\mu = 0$ or $\nu = 0\}$ $\delta^a{}_\mu = 0$ if $\{\mu = 0\}$ $\delta^a{}_\nu = 0$ if $\{\nu = 0\}$

Explicitly, $\eta^1_{\mu\nu} = \left(\begin{array}{c|ccc} 0 & -1 & 0 & 0 \\ \hline 1 & 0 & & \\ 0 & & 0 & 1 \\ 0 & & -1 & 0 \end{array}\right)$ $\eta^2_{\mu\nu} = \left(\begin{array}{c|ccc} 0 & 0 & -1 & 0 \\ \hline 0 & 0 & & -1 \\ 1 & & 0 & \\ 0 & 1 & & 0 \end{array}\right)$ $\eta^3_{\mu\nu} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 1 & \\ 0 & -1 & 0 & \\ 1 & & & 0 \end{array}\right)$

Properties:

$\eta^a{}_{\mu\nu} = \frac{1}{2}\hat{\epsilon}_{\mu\nu\rho\sigma}\eta^a{}_{\rho\sigma}$ (self-dual) $\hat{\epsilon}_{0123} = -1$
$(\hat{\epsilon}_{1230} = +1)$

$\eta^a{}_{\mu\nu} = -\eta^a{}_{\nu\mu}$ (antisymmetric)

$\eta^a{}_{\mu\nu}\eta^b{}_{\mu\nu} = 4\delta^{ab}$

$$\left.\begin{array}{l} \eta^a{}_{\mu\nu}\eta^a{}_{\mu\rho} = 3\delta_{\nu\rho} \\ \\ \eta^a{}_{\mu\nu}\eta^a{}_{\mu\nu} = 12 \end{array}\right\}$$

Proofs:
Leave as exercises

$\eta^a{}_{\mu\nu}\eta^b{}_{\mu\rho} = \delta^{ab}\delta_{\nu\rho} + \epsilon^{abc}\eta^c{}_{\nu\rho}$

Interesting to note:

If we let indices μ & ν to run $\{1,2,3,4\}$ instead of $\{0,1,2,3\}$ we would get:

$\eta^1 = \left(\begin{array}{ccc|c} & & & 1 \\ & & 1 & \\ & -1 & & \\ \hline -1 & & & \end{array}\right)$ $\eta^2 = \left(\begin{array}{ccc|c} & & -1 & \\ & & & 1 \\ 1 & & & \\ \hline & -1 & & \end{array}\right)$ $\eta^3 = \left(\begin{array}{ccc|c} & 1 & & \\ -1 & & & \\ & & & 1 \\ \hline & & -1 & \end{array}\right)$

These are precisely $(2i$ times$)$ the generators of custodial weak isospin in the Higgs sector of the SM: $(T^a_{cus}) = \frac{1}{2i}(\eta^a)$

Solution to exercises:

$$\eta^a{}_{\mu\nu}\eta^a{}_{\mu\rho} = \left(\epsilon^a{}_{\mu\nu} + \delta^a{}_\mu\delta_{\nu 0} - \delta^a{}_\nu\delta_{\mu 0}\right)\left(\epsilon^a{}_{\mu\rho} + \delta^a{}_\mu\delta_{\rho 0} - \delta^a{}_\rho\delta_{\mu 0}\right)$$

$\epsilon \equiv$ antisymmetric

$$= \overset{①}{\epsilon^a{}_{\mu\nu}\epsilon^a{}_{\mu\rho}} + \cancel{\epsilon^a{}_{\mu\nu}\delta^a{}_\mu\delta_{\rho 0}} - \underbrace{\epsilon^a{}_{\mu\nu}\delta^a{}_\rho\delta_{\mu 0}}_{0}$$

0 because $\epsilon^a{}_{0\nu} = 0$

$$+ \cancel{\delta^a{}_\mu\delta_{\nu 0}\epsilon^a{}_{\mu\rho}} + \overset{②}{\underbrace{\delta^a{}_\mu\delta_{\nu 0}\delta^a{}_\mu\delta_{\rho 0}}_{3}} - \underbrace{\delta^a{}_\mu\delta_{\nu 0}\delta^a{}_\rho\delta_{\mu 0}}_{0}$$

$$- \underbrace{\delta^a{}_\nu\delta_{\mu 0}\epsilon^a{}_{\mu\rho}}_{0} - \underbrace{\delta^a{}_\nu\delta_{\mu 0}\delta^a{}_\mu\delta_{\rho 0}}_{0} + \overset{③}{\underbrace{\delta^a{}_\nu\delta_{\mu 0}\delta^a{}_\rho\delta_{\mu 0}}_{1}}$$

① $\epsilon^a{}_{\mu\nu}\epsilon^a{}_{\mu\rho} = 2(\delta_{\nu\rho} - \delta_{\nu 0}\delta_{\rho 0})$ ← like $\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl}$, but must remove $\delta_{(\nu=0)(\rho=0)}$ contribution since $\epsilon^a{}_{\mu\nu} = 0$ for $\mu = 0$ or $\nu = 0$.

② $\delta^a{}_\mu\delta_{\nu 0}\delta^a{}_\mu\delta_{\rho 0} = 3\,\delta_{\nu 0}\delta_{\rho 0}$

③ $\delta^a{}_\nu\underbrace{\delta_{\mu 0}\delta^a{}_\rho\delta_{\mu 0}}_{1} = \delta^a{}_\nu\delta^a{}_\rho = \delta_{\nu\rho} - \delta_{\nu 0}\delta_{\rho 0}$ ← again, must remove $\delta_{(\nu=0)(\rho=0)}$ contribution.

So,

$$\eta^a{}_{\mu\nu}\eta^a{}_{\mu\rho} = 2(\delta_{\nu\rho} - \delta_{\nu 0}\delta_{\rho 0}) + 3\,\delta_{\nu 0}\delta_{\rho 0} + \delta_{\nu\rho} - \delta_{\nu 0}\delta_{\rho 0}$$

cancel

$$= 3\,\delta_{\nu\rho} \quad ▨$$

Then, setting $\nu = \rho$ and summing gives $\delta_{\rho\rho} = 4$, so that

$$\eta^a{}_{\mu\nu}\eta^a{}_{\mu\nu} = 3 \times 4 = 12 \quad ▨$$

So, in terms of 't Hooft symbols, we have at the surface,

$$A_\mu(|x| \to \infty) = \frac{-i}{g}\left(\delta_{\nu\rho} + i\eta^a{}_{\nu\rho}\sigma^a\right)\hat{x}_\nu \left(\frac{\delta_{\mu\rho} - \hat{x}_\mu \hat{x}_\rho}{|x|}\right)$$

$$= \frac{-i}{g}\frac{1}{|x|}\Big(\cancel{\hat{x}_\mu - \hat{x}_\nu \underbrace{\hat{x}_\mu \hat{x}_\nu}_{=1}} + i\eta^a{}_{\nu\mu}\hat{x}_\nu\sigma^a - i\underbrace{\eta^a{}_{\nu\rho}}_{\substack{\text{antisymm} \\ \nu\leftrightarrow\rho}}\underbrace{\hat{x}_\nu \hat{x}_\mu \hat{x}_\rho}_{\substack{\text{symm} \\ \nu\leftrightarrow\rho}}\sigma^a\Big)$$

$$= \frac{-i}{g}\frac{1}{|x|}\left(-i\eta^a{}_{\mu\nu}\hat{x}_\nu\sigma^a\right)$$

Write as $x_\nu/|x|$

$$A_\mu \equiv A^a_\mu(|x|\to\infty)\frac{\sigma^a}{2} = \frac{-1}{g}\frac{1}{x_E^2}\eta^a{}_{\mu\nu}x_\nu\sigma^a$$

Dropping the σ^a, we have

$$\boxed{A^a_\mu(|x|\to\infty) = \frac{-2}{g}\frac{1}{x_E^2}\eta^a{}_{\mu\nu}x_\nu}$$ on the boundary 3-sphere.

Using this result as a guide, we can write down an ansatz that would be valid for all (Euclidean) space-time — not just on the hypersphere:

<u>Ansatz:</u>

$$A^a_\mu(x) = \frac{-2}{g}\frac{f(x^2)}{x^2}\eta^a{}_{\mu\nu}x_\nu,$$

$f(x^2)$ is a function to be solved for satisfying $f(\infty)\to 1$, $f(0)\to 0$ (for continuous solutions)

Construct the field-strength tensor:

$$F^a_{\mu\nu} = \overset{①}{\partial_\mu A^a_\nu} - \overset{②}{\partial_\nu A^a_\mu} - \overset{③}{g f^{abc}A^b_\mu A^c_\nu}$$

① $$\partial_\mu A^a_\nu = \frac{-2}{g}\eta^a{}_{\nu\rho}\,\partial_\mu\left(\frac{f(x^2)x_\rho}{x^2}\right)$$ use quotient rule.

$$= \frac{-2}{g}\eta^a{}_{\nu\rho}\frac{1}{x^4}\left[\left(f'(x^2)\,\underline{\underline{2}}x_\mu x_\rho + f(x^2)\delta_{\mu\rho}\right)x^2 - f(x^2)x_\rho\,\underline{\underline{2}}x_\mu\right]$$

Factor out 2:

$$= \frac{-4}{g}\left[\frac{1}{x^4}\left(x^2 f'(x^2) - f(x^2)\right)\eta^a{}_{\nu\rho}x_\rho x_\mu + \frac{1}{x^2}f(x^2)\,\frac{1}{2}\,\eta^a{}_{\nu\mu}\right]$$

Similarly,

$$② \quad \partial_\nu A^a_\mu = \frac{-4}{g}\left[\frac{1}{x^4}\left(x^2 f'(x^2) - f(x^2)\right)\eta^a{}_{\mu\rho}x_\rho x_\nu + \frac{1}{x^2}f(x^2)\frac{1}{2}\eta^a{}_{\mu\nu}\right]$$

Then

$$\partial_\mu A^a_\nu - \partial_\nu A^a_\mu = \frac{-4}{g}\left[\frac{1}{x^4}\left(x^2 f'(x^2) - f(x^2)\right)\left(\eta^a{}_{\nu\rho}x_\rho x_\mu - \eta^a{}_{\mu\rho}x_\rho x_\nu\right) + \frac{1}{x^2}f(x^2)\frac{1}{2}\left(\eta^a{}_{\nu\mu} - \eta^a{}_{\mu\nu}\right)\right.$$

$$= \frac{-4}{g}\left[\frac{1}{x^4}\left(x^2 f'(x^2) - f(x^2)\right)\left(\eta^a{}_{\nu\rho}x_\rho x_\mu - \eta^a{}_{\mu\rho}x_\rho x_\nu\right) - \frac{1}{x^2}f(x^2)\eta^a{}_{\mu\nu}\right]$$

Now

$$③ \quad -g f^{abc}A^b_\mu A^c_\nu = g f^{abc}\left(\frac{-2}{g}\frac{f(x^2)}{x^2}\right)^2 \eta^b{}_{\mu\rho}\eta^c{}_{\nu\sigma}x_\rho x_\sigma$$

$$= \frac{-4}{g}\frac{f^2(x^2)}{x^4}\underbrace{f^{abc}\eta^b{}_{\mu\rho}\eta^c{}_{\nu\sigma}}x_\rho x_\sigma$$

Assume instanton solution in SU(2) subgroup: $f^{abc} \to \epsilon^{abc}$

$$= \frac{-4}{g}\frac{f^2(x^2)}{x^4}\left(\delta_{\mu\nu}\eta^a{}_{\rho\sigma} - \delta_{\mu\sigma}\eta^a{}_{\rho\nu} - \delta_{\rho\sigma}\eta^a{}_{\mu\nu} - \delta_{\rho\nu}\eta^a{}_{\mu\sigma}\right)x_\rho x_\sigma$$

$$= \frac{-4}{g}\frac{f^2(x^2)}{x^4}\left(-\eta^a{}_{\rho\nu}x_\rho x_\mu + \eta^a{}_{\mu\nu}x^2 - \eta^a{}_{\mu\sigma}x_\sigma x_\nu\right)$$

rename $\sigma \to \rho$

$$= \frac{-4}{g}\left[\frac{-f^2(x^2)}{x^2}\eta^a{}_{\mu\nu} + \frac{1}{x^4}f^2(x^2)\left(\eta^a{}_{\nu\rho}x_\rho x_\mu - \eta^a{}_{\mu\rho}x_\rho x_\nu\right)\right]$$

So, adding everything together,

$$F^a_{\mu\nu} = \frac{-4}{g}\left[\frac{1}{x^4}\left(x^2 f'(x^2) - f(x^2) + f^2(x^2)\right)\left(\eta^a{}_{\nu\rho}x_\rho x_\mu - \eta^a{}_{\mu\rho}x_\rho x_\nu\right) + \frac{1}{x^2}\left(-f(x^2) + f^2(x^2)\right)\eta^a{}_{\mu\nu}\right]$$

$$= \frac{-4}{g}\left[\frac{x^2 f' - f(1-f)}{x^4}\left(\eta^a{}_{\nu\rho}x_\rho x_\mu - \eta^a{}_{\mu\rho}x_\rho x_\nu\right) - \frac{f(1-f)}{x^2}\eta^a{}_{\mu\nu}\right]$$