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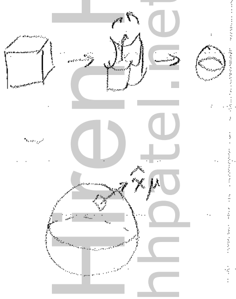
Could plug into Euclid. action, and extremize with respect to $f(x^2)$. $\longrightarrow$ Tedious || Exercise

Better to derive a <u>Bogomolny bound</u>:

Recall the winding number:

$$\nu = \frac{-1}{24\pi^2}\int d^3x\, \epsilon_{ijk}\, \mathrm{Tr}\left[(U\partial_i U^\dagger)(U\partial_j U^\dagger)(U\partial_k U^\dagger)\right]$$ (checked by direct integration)



Write integral over 3-volume as an integral over the surface of the 3-sphere

$$\nu = \frac{+1}{24\pi^2}\int d\hat{x}_\mu\, \hat{\epsilon}_{\mu\nu\rho\sigma}\, \mathrm{Tr}\left[(U\partial_\nu U^\dagger)(U\partial_\rho U^\dagger)(U\partial_\sigma U^\dagger)\right]$$ (overall minus sign from ϵ-symbol)

Recall that $A_\mu = \frac{-i}{g} U\partial_\mu U^\dagger$ since it is a gauge transform of 0.

$$\nu = \frac{1}{24\pi^2}\int d\hat{x}_\mu\, \hat{\epsilon}_{\mu\nu\rho\sigma}\left(\frac{g}{-i}\right)^3 \mathrm{Tr}[\hat{A}_\nu \hat{A}_\rho \hat{A}_\sigma]$$

$$= \frac{g^2}{16\pi^2}\int d\hat{x}_\mu\, \hat{\epsilon}_{\mu\nu\rho\sigma}\, \frac{-2}{3}\, ig\, \mathrm{Tr}[\hat{A}_\nu \hat{A}_\rho \hat{A}_\sigma]$$

Since $F_{\mu\nu} = 0$ at the surface, can freely add it:

$$= \frac{g^2}{16\pi^2}\int d\hat{x}_\mu\, \hat{\epsilon}_{\mu\nu\rho\sigma}\, \mathrm{Tr}\left[\hat{A}_\nu \hat{F}_{\rho\sigma} - \frac{2}{3} ig\, \hat{A}_\nu \hat{A}_\rho \hat{A}_\sigma\right]$$

Integrate by parts (use Gauss' theorem) to convert surface integral into 4-volume integral of a divergence:

$$\nu = \frac{g^2}{16\pi^2}\int d^4x_E\, \underbrace{\hat{\epsilon}_{\mu\nu\rho\sigma}\, \partial_\mu\, \mathrm{Tr}\left[\hat{A}_\nu \hat{F}_{\rho\sigma} - \frac{2}{3} ig\, \hat{A}_\nu \hat{A}_\rho \hat{A}_\sigma\right]}_{\frac{1}{2}\partial_\mu K_\mu = \frac{1}{2}\hat{\epsilon}_{\mu\nu\rho\sigma}\mathrm{Tr}[\hat{F}_{\mu\nu}\hat{F}_{\rho\sigma}]}$$ — see anomalies

$K_\mu \equiv$ Chern-Simons current density

$$\nu = \frac{g^2}{32\pi^2}\int d^4x_E\, \hat{\epsilon}_{\mu\nu\rho\sigma}\, \mathrm{Tr}[\hat{F}_{\mu\nu}\hat{F}_{\rho\sigma}]$$

$$= \frac{g^2}{16\pi^2}\int d^4x_E\, \mathrm{Tr}[\tilde{F}_{\mu\nu} F_{\mu\nu}], \qquad \tilde{F}_{\mu\nu} = \frac{1}{2}\hat{\epsilon}_{\mu\nu\rho\sigma} F^{\rho\sigma}$$

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Set $t = \text{const}$, and identify the surface at infinity ($|\vec{x}| \to \infty$) with one point.
$\Rightarrow$ Result: A 3-sphere of infinite radius. — Parametrize by three angles: (θ, ϕ, χ)

In[1]:=

$$U = \begin{pmatrix} \mathtt{Cos}[\chi] + i\,\mathtt{Sin}[\chi]\,\mathtt{Cos}[\theta] & i\,\mathtt{Sin}[\chi]\,\mathtt{Sin}[\theta]\,e^{-i n \phi} \\ i\,\mathtt{Sin}[\chi]\,\mathtt{Sin}[\theta]\,e^{i n \phi} & \mathtt{Cos}[\chi] - i\,\mathtt{Sin}[\chi]\,\mathtt{Cos}[\theta] \end{pmatrix}$$

$$U^{\dagger} = \mathtt{FullSimplify}\left[\mathtt{Conjugate}\left[\mathtt{Transpose}\left[\begin{pmatrix} \mathtt{Cos}[\chi] + i\,\mathtt{Sin}[\chi]\,\mathtt{Cos}[\theta] & i\,\mathtt{Sin}[\chi]\,\mathtt{Sin}[\theta]\,e^{-i n \phi} \\ i\,\mathtt{Sin}[\chi]\,\mathtt{Sin}[\theta]\,e^{i n \phi} & \mathtt{Cos}[\chi] - i\,\mathtt{Sin}[\chi]\,\mathtt{Cos}[\theta] \end{pmatrix}\right]\right],\right.$$
$$\left.\mathtt{Assumptions} \to \{\chi \in \mathtt{Reals}, \phi \in \mathtt{Reals}, \theta \in \mathtt{Reals}, n \in \mathtt{Integers}\}\right]$$

Out[1]= $\{\{\mathtt{Cos}[\chi] + i\,\mathtt{Cos}[\theta]\,\mathtt{Sin}[\chi],\ i\,e^{-i n \phi}\,\mathtt{Sin}[\theta]\,\mathtt{Sin}[\chi]\},\ \{i\,e^{i n \phi}\,\mathtt{Sin}[\theta]\,\mathtt{Sin}[\chi],\ \mathtt{Cos}[\chi] - i\,\mathtt{Cos}[\theta]\,\mathtt{Sin}[\chi]\}\}$

Out[2]= $\{\{\mathtt{Cos}[\chi] - i\,\mathtt{Cos}[\theta]\,\mathtt{Sin}[\chi],\ -i\,e^{-i n \phi}\,\mathtt{Sin}[\theta]\,\mathtt{Sin}[\chi]\},\ \{-i\,e^{i n \phi}\,\mathtt{Sin}[\theta]\,\mathtt{Sin}[\chi],\ \mathtt{Cos}[\chi] + i\,\mathtt{Cos}[\theta]\,\mathtt{Sin}[\chi]\}\}$

In[3]:=
```
UχU† = U.D[U†, χ] // Simplify;
UθU† = U.D[U†, θ] // Simplify;
UϕU† = U.D[U†, ϕ] // Simplify;
```

Now the six permutations that would be induced by ϵ_{ijk}: $\epsilon_{\chi\theta\phi}, \epsilon_{\theta\phi\chi}, \epsilon_{\phi\chi\theta}$ and $\epsilon_{\theta\chi\phi}, \epsilon_{\chi\phi\theta}, \epsilon_{\phi\theta\chi}$.

In[6]:=
```
UχU†.UθU†.UϕU† // Simplify
UθU†.UϕU†.UχU† // Simplify
UϕU†.UχU†.UθU† // Simplify
```

Out[6]= $\{\{-n\,\mathtt{Sin}[\theta]\,\mathtt{Sin}[\chi]^2, 0\}, \{0, -n\,\mathtt{Sin}[\theta]\,\mathtt{Sin}[\chi]^2\}\}$

Out[7]= $\{\{-n\,\mathtt{Sin}[\theta]\,\mathtt{Sin}[\chi]^2, 0\}, \{0, -n\,\mathtt{Sin}[\theta]\,\mathtt{Sin}[\chi]^2\}\}$

Out[8]= $\{\{-n\,\mathtt{Sin}[\theta]\,\mathtt{Sin}[\chi]^2, 0\}, \{0, -n\,\mathtt{Sin}[\theta]\,\mathtt{Sin}[\chi]^2\}\}$

In[9]:=
```
UθU†.UχU†.UϕU† // Simplify
UχU†.UϕU†.UθU† // Simplify
UϕU†.UθU†.UχU† // Simplify
```

Out[9]= $\{\{n\,\mathtt{Sin}[\theta]\,\mathtt{Sin}[\chi]^2, 0\}, \{0, n\,\mathtt{Sin}[\theta]\,\mathtt{Sin}[\chi]^2\}\}$

Out[10]= $\{\{n\,\mathtt{Sin}[\theta]\,\mathtt{Sin}[\chi]^2, 0\}, \{0, n\,\mathtt{Sin}[\theta]\,\mathtt{Sin}[\chi]^2\}\}$

Out[11]= $\{\{n\,\mathtt{Sin}[\theta]\,\mathtt{Sin}[\chi]^2, 0\}, \{0, n\,\mathtt{Sin}[\theta]\,\mathtt{Sin}[\chi]^2\}\}$

Hence, the winding number of this map is

In[12]:= $-\dfrac{1}{24\,\pi^2}\,\mathtt{Integrate}[6 * \mathtt{Tr}[U\chi U^{\dagger}.U\theta U^{\dagger}.U\phi U^{\dagger}\ //\ \mathtt{Simplify}], \{\chi, 0, \pi\}, \{\theta, 0, \pi\}, \{\phi, 0, 2\pi\}]$

Out[12]= n

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Having expressed the winding number as a 4d volume integral over gauge fields, we can derive the Bogomolny bound:

Note: $\frac{1}{2}\mathrm{Tr}[\tilde{F}_{\mu\nu} \pm F_{\mu\nu}]^2 = \mathrm{Tr}[F_{\mu\nu}F_{\mu\nu}] \pm \mathrm{Tr}[\tilde{F}_{\mu\nu}F_{\mu\nu}]$

since $\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu} = F_{\mu\nu}F_{\mu\nu}$

Since LHS is non-negative, RHS must also be nonnegative:

$$\mathrm{Tr}[F_{\mu\nu}F_{\mu\nu}] \pm \mathrm{Tr}[\tilde{F}_{\mu\nu}F_{\mu\nu}] \geq 0$$

$$\Rightarrow \underbrace{\int d^4x\, \mathrm{Tr}[F_{\mu\nu}F_{\mu\nu}]}_{\text{This is } 2\times S_E[A]} \geq \mp \underbrace{\int d^4x\, \mathrm{Tr}[\tilde{F}_{\mu\nu}F_{\mu\nu}]}_{\text{This is } 16\pi^2 n/g^2}$$

So, $$2\, S_E[A] \geq \mp \frac{16\pi^2 n}{g^2} = \frac{16\pi^2 |n|}{g^2}$$

$$\Rightarrow \boxed{S_E[A] \geq \frac{8\pi^2 |n|}{g^2}} \quad \text{Bogomolny bound.}$$

Note that the bound is saturated if

$\tilde{F}_{\mu\nu} = +F_{\mu\nu}$ for $n>0$ (self-dual)

$\tilde{F}_{\mu\nu} = -F_{\mu\nu}$ for $n<0$. (anti-self dual)

Assuming that the instanton solution saturates the Bogomolny bound, let us proceed to solve for $f(x^2)$:

$$\tilde{F}^a_{\mu\nu} = \frac{1}{2}\hat{\epsilon}_{\mu\nu\rho\sigma} F_{\rho\sigma}$$

$$= \frac{-4}{g}\left[\frac{x^2 f' - f(1-f)}{x^4}\underbrace{\frac{1}{2}\hat{\epsilon}_{\mu\nu\rho\sigma}(\eta^a{}_{\sigma\lambda}x_\lambda x_\rho - \eta^a{}_{\rho\lambda}x_\lambda x_\sigma)}_{\hat{\epsilon}_{\mu\nu\rho\sigma}\eta^a{}_{\sigma\lambda}x_\lambda x_\rho \;=\; ①} - \frac{f(1-f)}{x^2}\underbrace{\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\eta^a{}_{\rho\sigma}}_{②}\right]$$

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① $\epsilon_{\mu\nu\rho\sigma}\,\eta^a_{\sigma\lambda}\, x_\lambda x_\rho = \left(-\delta_{\lambda\mu}\eta^a_{\nu\rho} - \delta_{\lambda\rho}\eta^a_{\mu\nu} - \delta_{\lambda\nu}\eta^a_{\rho\mu}\right) x_\lambda x_\rho$

$$= -\left(\eta^a_{\nu\rho}\, x_\rho x_\mu + \eta^a_{\mu\nu}\, x^2 - \eta^a_{\mu\rho}\, x_\rho x_\nu\right)$$

② $\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\,\eta^a_{\rho\sigma} = \frac{1}{2}\left(\eta^a_{\nu\mu} + 4\eta^a_{\mu\nu} + \eta^a_{\nu\mu}\right)$

$$= \frac{1}{2}\, 2\eta^a_{\mu\nu} = \eta^a_{\mu\nu}$$ (t' Hooft symbols self-dual)

So,

$$\tilde{F}^a_{\mu\nu} = \frac{-4}{g}\left[\frac{x^2 f' - f(1-f)}{x^4} \times -\left(\eta^a_{\nu\rho}x_\rho x_\mu + \eta^a_{\mu\nu}x^2 - \eta^a_{\mu\rho}x_\rho x_\nu\right) - \frac{f(1-f)}{x^2}\eta^a_{\mu\nu}\right]$$

add (partially cancel)

$$= \frac{-4}{g}\left[-\frac{x^2 f' - f(1-f)}{x^4}\left(\eta^a_{\nu\rho}x_\rho x_\mu - \eta^a_{\mu\rho}x_\rho x_\nu\right) - f'\eta^a_{\mu\nu}\right]$$

Setting $F^a_{\mu\nu} = \tilde{F}^a_{\mu\nu}$ (saturation of Bogomolny bound), equating either term gives an ordinary 1st order ODE:

$$x^2 f' - f(1-f) = 0$$ subject to $f(x^2 \to \infty) = 1$, $f(x^2 \to 0) = 0$ (for continuity)

Solution: $$f(x^2) = \frac{x^2}{x^2 + \rho^2}$$ constant of integration (size of instanton)

So, the Belavin-Polyakov-Schwartz-Tyupkin instanton ($n = +1$) solution is

$$\boxed{A^a_\mu(x)^{\text{inst}} = \frac{-2}{g}\,\frac{\eta^a_{\mu\nu}x_\nu}{x^2+\rho^2}} \quad \Rightarrow \quad F^a_{\mu\nu}(x)_{\text{inst}} = \frac{4}{g}\,\frac{\rho^2}{(x^2+\rho^2)^2}\,\eta^a_{\mu\nu}$$

Since solution saturates Bogomolny bound, the Euclidean action for the instanton is

$$S_E[A^{\text{inst}}] = \frac{8\pi^2}{g^2}. \quad \Rightarrow \quad \langle n+1|\,\hat{H}_{QCD}\,|n\rangle \sim e^{-8\pi^2/g^2}$$