<u>BPST Instanton by Solving Equation of Motion</u> — Solution to exercise.

<u>Equation of motion:</u>

$$\left(D_\mu^{ab} F_{\mu\nu}^{b}\right)_E = 0$$

$$D_\mu^{ab} = \partial_\mu \delta^{ab} + g\epsilon^{abc} A_\mu^c, \qquad A_\mu^c = \frac{-2}{g}\frac{f}{x^2}\eta^c{}_{\mu\sigma} x_\sigma \quad ; \quad f = f(x^2)$$

$$= \partial_\mu \delta^{ab} + g\,\epsilon^{abc}\frac{-2}{g}\frac{f}{x^2}\eta^c{}_{\mu\sigma} x_\sigma$$

$$= \partial_\mu \delta^{ab} - 2f\frac{x_\sigma}{x^2}\epsilon^{abc}\eta^c{}_{\mu\sigma}$$

$$F_{\mu\nu}^b = \frac{-4}{g}\left[\frac{x^2 f' - f(1-f)}{x^4}\left(\eta^b{}_{\nu\rho}x_\rho x_\mu - \eta^b{}_{\mu\rho}x_\rho x_\nu\right) - \frac{f(1-f)}{x^2}\eta^b{}_{\mu\nu}\right]$$

So, the equation of motion becomes:

$$D_\mu^{ab}F_{\mu\nu}^b = \left(\underline{\partial_\mu \delta^{ab}} - 2f\frac{x_\sigma}{x^2}\epsilon^{abc}\eta^c{}_{\mu\sigma}\right)F^b_{\mu\nu} = 0$$

Look at <u>derivative term</u> first:

$$\partial_\mu \delta^{ab} F_{\mu\nu}^b = \frac{-4}{g}\left[\partial_\mu\left(\frac{x^2 f' - f(1-f)}{x^4}\right)\left(\eta^a{}_{\nu\rho}x_\rho x_\mu - \eta^a{}_{\mu\rho}x_\rho x_\nu\right)\right.$$
$$+ \frac{x^2 f' - f(1-f)}{x^4}\,\partial_\mu\left(\eta^a{}_{\nu\rho}x_\rho x_\mu - \eta^a{}_{\mu\rho}x_\rho x_\nu\right)$$
$$\left. + \partial_\mu\left(\frac{-f(1-f)}{x^2}\right)\eta^a{}_{\mu\nu}\right]$$

$$= \frac{-4}{g}\left\{\frac{1}{x^8}\left[\left(2x_\mu f' + x^2 f'' 2x_\mu - f' 2x_\mu(1-f) - f(-f' 2x_\mu)\right)x^4\right.\right.$$
$$\left. - \left(x^2 f' - f(1-f)\right)2x^2\cdot 2x_\mu\right]\left(\eta^a{}_{\nu\rho}x_\rho x_\mu - \eta^a{}_{\mu\rho}x_\rho x_\nu\right)$$
$$+ \frac{x^2 f' - f(1-f)}{x^4}\left(\eta^a{}_{\nu\mu}x_\mu + \eta^a{}_{\nu\rho}x_\rho\delta_{\mu\mu} - \cancel{\eta^a{}_{\mu\rho}\delta^\mu{}_\rho} x_\nu - \eta^a{}_{\mu\rho}x_\rho \delta\right.$$
$$\left. - \frac{1}{x^4}\left[\left(f' 2x_\mu(1-f) + f(-f' 2x_\mu)\right)x^2 + f(1-f)2x_\mu\right]\eta^a{}_{\mu\nu}\right\}$$

In first term, x_μ factors $\rightarrow \eta^a{}_{\nu\rho}x_\rho x_\mu \cdot x_\mu = x^2\eta^a{}_{\nu\rho}x_\rho$; but $-\eta^a{}_{\mu\rho}x_\rho x_\nu x_\mu$ vanishes.

In second term, $\cancel{\eta^a{}_{\nu\mu}x_\mu} - \eta^a{}_{\nu\rho}x_\rho\delta_{\mu\mu} - 0 - \cancel{\eta^a{}_{\mu\rho}x_\rho}\delta^\mu{}_\nu = 4\eta^a{}_{\nu\rho}x_\rho$

$$\partial_\mu \delta^{ab} F^b_{\mu\nu} = \frac{-4}{g}\Big\{ \frac{1}{x^{\cancel{8}\,6\,4}} \Big[\big(2f' + x^2 f''\cdot 2 - f'2(1-f) + f(f'2)\big) x^{\cancel{4}\,2}$$

$$- \big(x^2 f' - f(1-f)\big)\, 2\cancel{x^2}\cdot 2 \Big] \cancel{x^2}\, \eta^a{}_{\nu\rho}\, x_\rho$$

$$+ \frac{x^2 f' - f(1-f)}{x^4} \cdot 4\, \eta^a{}_{\nu\rho}\, x_\rho$$

$$- \frac{1}{x^4}\Big[\big(f'2(1-f) - f(f'\cdot 2)\big)x^2 + f(1-f)2\Big]\, \eta^a{}_{\mu\nu}\, x_\mu \Big\}$$

ch. index $\mu \to \rho$

$$= \frac{-4}{g}\,\frac{1}{x^4}\cdot 2\Big[x^4 f'' + x^2 f' - (1-f)f\Big]\, \eta^a{}_{\nu\rho}\, x_\rho$$

$$= \frac{-8}{g}\,\frac{1}{x^4}\Big[x^4 f''(x^2) + x^2 f'(x^2) - \big(1 - f(x^2)\big) f(x^2)\Big]\, \eta^a{}_{\nu\rho}\, x_\rho$$

Now look at ϵ^{abc} term:

$$-2f\,\frac{x_\sigma}{x^2}\,\epsilon^{abc}\,\eta^c{}_{\mu\sigma} \times \frac{-4}{g}\Big[\frac{x^2 f' - f(1-f)}{x^4}\big(\eta^b{}_{\nu\rho}\, x_\rho x_\mu - \eta^b{}_{\mu\rho}\, x_\rho x_\nu\big) - \frac{f(1-f)}{x^2}\,\eta^b{}_{\mu\nu}\Big]$$

$$= \frac{+8}{g}\,\frac{1}{x^4}\,\frac{1}{x^2}\, f\big(x^2 f' - f(1-f)\big)\,\epsilon^{abc}\Big(\eta^b{}_{\nu\rho}\,\underbrace{\eta^c{}_{\mu\sigma}\, x_\rho x_\mu x_\sigma}_{\text{vanishes by symm}} - \underbrace{\eta^b{}_{\mu\rho}\,\eta^c{}_{\mu\sigma}\, x_\rho x_\nu x_\sigma}_{\text{I}}\Big)$$

$$- \frac{8}{g}\,\frac{f^2(1-f)}{x^4}\,\epsilon^{abc}\,\underbrace{\eta^b{}_{\mu\nu}\,\eta^c{}_{\mu\sigma}\, x_\sigma}_{\text{II}}$$

I

$$-\epsilon^{abc}\,\eta^b{}_{\mu\rho}\,\eta^c{}_{\mu\sigma}\, x_\rho x_\nu x_\sigma = -\underbrace{\epsilon^{abc}\big(\delta^{bc}}_{\text{vanishes by symmetry}}\delta_{\rho\sigma} + \epsilon^{bcd}\,\eta^d{}_{\rho\sigma}\big)\underbrace{x_\rho x_\nu x_\sigma}_{\text{vanishes by symmetry}} = 0$$

II

$$\epsilon^{abc}\,\eta^b{}_{\mu\nu}\,\eta^c{}_{\mu\sigma}\, x_\sigma = \underbrace{\epsilon^{abc}\big(\delta^{bc}}_{\text{vanishes by sym}}\delta_{\nu\sigma} + \epsilon^{bcd}\,\eta^d{}_{\nu\sigma}\big)\, x_\sigma$$

$$= \epsilon^{abc}\,\epsilon^{bcd}\,\eta^d{}_{\nu\sigma}\, x_\sigma = 2\delta^{ad}\,\eta^d{}_{\nu\sigma}\, x_\sigma = 2\eta^a{}_{\nu\rho}\, x_\rho$$

rename $\sigma \to \rho$

So, ϵ^{abc} term becomes

$$= \frac{-8}{g}\,\frac{2f^2(1-f)}{x^4}\,\eta^a{}_{\nu\rho}\, x_\rho$$

So, our equation of motion is:

$$D_\mu^{ab} F_{\mu\nu}^{b} = \frac{-8}{g}\frac{1}{x^4}\left[x^4 f'' + x^2 f' - f(1-f)\right]\eta^a_{\nu\rho}x_\rho - \frac{8}{g}\frac{2f^2(1-f)}{x^4}\eta^a_{\nu\rho}x_\rho$$

$$= \frac{-8}{g}\frac{1}{x^4}\left[\left(x^4 f'' + 2f^2(1-f)\right) + \left(x^2 f' - f(1-f)\right)\right]\eta^a_{\nu\rho}x_\rho = 0.$$

So, the equation of motion for $f(x^2)$ is:

$$\left(x^4 f'' + 2f^2(1-f)\right) + \left(x^2 f' - f(1-f)\right) = 0$$

Solution: $f(x^2) = \dfrac{x^2}{x^2+\rho^2}$ ← constant of integration: size of BPST instanton.

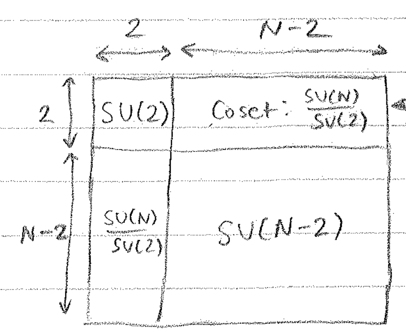
(Solution trivially verified: 1st term vanishes, 2nd term is the Diff. eq. derived by $F_{\mu\nu} = \tilde{F}_{\mu\nu}$ cond.)

Hence $\boxed{A_\mu^a(x)_{inst} = \dfrac{-2}{g}\dfrac{\eta^a_{\mu\nu}x_\nu}{x^2+\rho^2}}$ $\Rightarrow$ $F_{\mu\nu}^a(x)_{inst} = \dfrac{4}{g}\dfrac{\rho^2}{(x^2+\rho^2)^2}\eta^a_{\mu\nu}$

Zero Modes of the instanton

Translations:	3 space + 1 imaginary time
Scale:	1 dilatation
Orbital-isospin rotations:	3 rotations
Additional rotations ↓	$4(N-2)$ coset

If instanton occupies $SU(2)$ subgroup of bigger gauge group $SU(N)$:



The generators in the "coset" ← Area of rectangle $= 2 \times \underbrace{2(N-2)}_{\#\text{ of elements}}$

↑ Real & imaginary

generate new color-orientations of the instanton.

Total: $(3+1) + 1 + 3 + 4(N-2) = 4N$ zero-modes.